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THE INFLUENCE OF MOLECULAR STRUCTURE UPON THE INTERNAL FRICTION OF CERTAIN ISOMERIC ETHER GASES.[1]

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THE following investigation of the coefficients of friction of a group of isomeric ether gases, was undertaken with a view of determining whether the differences in molecular structure which are believed to exist in these isomeric compounds, are accompanied by differences in such a physical property as their internal friction.

It has been demonstrated[2] that the coefficient of friction of a gas

$$\eta = \frac{MG}{4\pi s^2}$$

where M is the mass of a molecule, G the mean value of the velocity as deduced from the mean kinetic energy, and s equals the distance between the center of two molecules at impact, or the diameter of the sphere of action. In isomeric compounds M is constant, and if the temperature is constant, G is also constant, so that if a difference in friction is observed, this difference must be due to a change in s, and, therefore, to a change in the size or the shape of the molecules owing to their different internal structure.

Investigations along this line were made by Meyer & Schumann,[3] in 1881. They determined the coefficient of internal friction by the well known transpiration method. Their apparatus consisted

[1] A paper read before the American Physical Society, March 2, 1907.

[2] O. E. Meyer's Kinetic Theory of Gases, 1899, p. 179.

[3] Wied. Ann., 1881, Bd. 13, p. 1.

essentially of a capillary nearly 1.5 meters in length coiled into a helix and fastened into the upper part of a boiling flask. The lower end of the capillary passed through the neck of the flask into a condenser. The substance to be examined was boiled at a regulated pressure, the vapor evolved surrounding the capillary and raising it to the same temperature. Part of the vapor which was, of course, saturated, entered the upper end of the capillary and passed through it into an air-free, cooled space where it was condensed and measured as a liquid. The volume of the vapor transpired was calculated from the amount of the liquid collected in the condenser.

Their method is open to criticism for several reasons. Objection might be made against the coiling of the capillary into a small helix because its bore may have thus been deformed and the curved path of the gas may have caused additional resistance. The vapor at the boiling point of the liquid can also hardly be regarded as a true gas, and should have been tested at several degrees above boiling. The expansion of the saturated vapor as it passed through the capillary probably also changed its state. Furthermore, the great length of time, several hours, needed for transpiration made it difficult to keep conditions constant.

TABLE V.

Esters $C_n H_{2n} O_2$ $\eta \times 10^6$.

Acid Radical.	Alcohol Radical.				
	Methyl.	Ethyl.	Propyl.	Isobutyl.	Amyl.
	$n=2$	3	4	5	6
Formic acid.	173	156	159	172	160
	$n=3$	4	5	6	7
Acetic acid.	152	152	160	155	
	$n=4$	5	6	7	8
Propionic acid.	150	158	153	164	158
	$n=5$	6	7	8	9
Normal butyric acid.	159	160	164	167	155
Isobutyric acid.	152	151	153	158	155
	$n=6$	7	8	9	10
Valerianic acid.	163	165	167	154	

Two capillaries were employed, one 1427 mm. long and 0.31 mm. in diameter; the other 1404 mm. long and 0.3328 mm. in diam-

eter. The second capillary gave values 3 per cent. higher than the first. Above is the table of their results for esters :

They conclude that all esters at their boiling points and at the same pressure transpire very nearly equal volumes of vapor, which, however, because the boiling points are different, do not contain the same number of molecules.

They found η for the corresponding acids considerably smaller than for their esters. They believed that the differences of η are partly too small and partly too irregular to draw any sure conclusion of the dependence of η on the molecular constitution. They point out, however, some differences which seem quite regular.

The esters of acetic, propionic and isobutyric acids show almost always a smaller η than those of formic, normal butyric and valerianic acids. This difference is especially noticeable in the two butyric acids. The influence of the alcohol radical cannot be seen so clearly. Among the isomeric esters, those of formic, acetic, propionic and isobutyric acids have the greater η with the greater alcohol radical. The isomeric esters of normal butyric and valerianic acids do not follow this rule but have the same friction.

The number of carbon atoms in a molecule also seems to have a certain influence on η. The esters for which $n = 2, 5, 7$ or 8 show a larger η than those with 3, 4 or 9 carbon atoms. The cause of these differences they were unable to explain.

On calculating the relative molecular volumes they found them only about half those given by Kopp's rule. L. Meyer tried to account for this by advancing the hypothesis that by Kopp's method the empty space is included which is open to the atoms for their motion, while from the coefficient of internal friction only the volume of the gas particles themselves is determined.

Steudel[1] continued the work of L. Meyer and Schumann, using the same apparatus with the second capillary, the first having been broken. He investigated several homologous lines of organic compounds, viz.: alcohols up to four atoms of carbon per molecule, and their halogen derivatives, also some substitution products of ethane and methane.

He found the transpiration time increased with the molecular

[1] Wied. Ann., 1882, Bd. 16, p. 369.

weight. Of isomeric compounds at the boiling points, the normal, *i. e.*, those that boil at the highest temperature transpire the slowest and the tertiary the fastest, with the exception of isopropyl alcohol which transpires noticeably slower than the normal. Unsymmetrical low boiling compounds have a smaller transpiration time than the symmetrical. The only exception he found to the rule that the transpiration time increases with the molecular weight is methyliodide whose time was 1,037 minutes, which is almost the same as 1,056 minutes taken by isobutyliodide.

He points out that the coefficients of friction of each line of homologous compounds are nearly alike or only slightly different. The values for the primary alcohols vary from .000135 to .000143; for three of them they are almost exactly alike. The isopropyl gave a considerably larger figure, also the tertiary butyl. The chlorides seem to all have the same friction. The greatest differences exist in the iodide column.

TABLE OF $\eta \times 10^6$.

Radical.	Alcohol.	Chloride.	Bromide.	Iodide.
Methyl.	135	116[1]		245
Ethyl.	142	105[1]	183	216
Normal propyl.	142	146	184	210
Isopropyl.	162	148	176	201
Normal butyl.	143	149		202
Isobutyl.	144	150	179	204
Tertiary butyl.	160	150		

He found a considerable difference in η with a change in pressure. With a greater pressure than 10 to 15 cm. of mercury the formula seemed to lose its validity for vapors. This agrees with what L. Meyer had noticed with benzole.

Steudel also gives tables of molecular velocities, mean free paths, combined cross-section of molecules and cross-sections of an equal number of molecules. He points out that the cross-sections of molecules of isomeric compounds are not the same. For the butyl compounds the areas for the normal substances are the largest, the tertiary the smallest, with the iso compounds between. Propyl and isopropyl alcohol, as also both chlorides show the same relation.

[1] Calculated from Graham's results.

On the other hand, isopropyl-bromide has a greater molecular area than the normal compound. In the substitution products of ethane the symmetrical ones have a larger area than the unsymmetrical.

Steudel also calculated the relative molecular volumes from SO_2 according to the method of L. Meyer, with whom he agrees in finding the volumes about one half as large as those calculated according to Kopp's rule. The ratios of molecular volumes of the different substances, however, are almost the same as the ratios of the volumes according to Kopp.

L. Meyer,[1] summing up Steudel's and his own previous results, gives the following:

TABLE OF $\eta \times 10^6$.

Alcohols	$C_nH_{2n+2}O$	142	average
Chlorides	$C_nH_{2n+1}Cl$	150	"
Esters	$C_nH_{2n}O_2$	155	"
Bromides	$C_nH_{2n+1}Br$	182	"
Iodides	$C_nH_{2n+1}I$	210	"

Cases in which $n = 1$ and some few unaccountable variations are omitted from the above. When $n = 1$ the variations are great but he could see no law of the influence of molecular constitution on the friction.

On the other hand the influence of the nature of the atom is very clearly seen; friction of iodine $>$ bromine $>$ chlorine.

The molecular volumes do not agree with those of L. Meyer[2] calculated from Graham's results. He points out that this one fact seems certain, viz.: the molecules of a tertiary butyl compound are smaller than those of a secondary which are smaller than those of a primary. This is in agreement with the universal view taken of the concatenation of these compounds. Those of the tertiary are grouped around a single atom of carbon, hence are more spherical in shape. Propyl and isopropyl compounds on the other hand do not show the same regularity. The alcohols and iodides deviate in opposite directions, which is unexplained. He further points out that the sphere of action of a liquid molecule increases with the temperature, while that of a gaseous molecule diminishes with rise of temperature.

[1] Pogg. Ann., 1882, Vol. 16, p. 394.

[2] Liebig's Ann., 1867, Suppl. Bd. 5, p. 129.

METHOD EMPLOYED IN THIS INVESTIGATION.

Owing to the difficulty of getting capillaries of perfectly uniform bore, and of determining with the greatest accuracy the shape and size of their cross-section, and also owing to the possible formation of eddies at the beginning and end of the flow, and to the possible slipping of the gas on the capillary walls, probably the oscillation method is the better for absolute measurements, provided a solid body is employed of such a shape that the mathematical treatment is rigorously correct. It seems certain however that the transpiration method has been growing in favor of late years, and that it is more convenient for comparative measurements than the other method. As my object was more to get comparative than absolute values of the greatest accuracy, the transpiration method was decided upon for this experimental investigation.

GENERAL DESCRIPTION OF APPARATUS.

The form of apparatus as well as the subject of this research was kindly suggested to me by Professor Morris Loeb, then of N. Y. University, under whose guidance these experiments were made. The apparatus consists essentially of a U-shaped tube (see Fig. 1), one limb of which is capillary, while the other is not, but serves as a cylinder of known capacity, down which is forced by gravity a piston consisting of a column of mercury, which drives the gas under it up through the capillary limb. The capillary, *A*, does not begin at the bend of the U, but some distance above it. The distance between the limbs is only 3 cm., so that the apparatus can be placed inside of a glass tube 5 cm. in diameter which serves as a steam jacket (see Fig. 3), the steam being introduced through a side tube, *S*, near the top and passing out freely at the bottom, *O*. A cork, *C*, through which both limbs of the apparatus pass is on a level with the top of the capillary, and serves to close the steam jacket at the top. The mercury limb above the cork is provided with a special stop-cock, *H*, bored out to the same diameter as the tube below it. Above the stop-cock is a continuation of the tube terminating in a small funnel, *F*, for convenience in introducing ether and mercury. An ordinary thistle funnel with horizontal bottom will not answer; the bottom of the funnel must be inclined

about 45° to the axis of the mercury tube, otherwise the mercury will not so readily descend in one unbroken column when released by turning the stop-cock. Two marks, M_1 and M_2, are etched on the mercury tube a convenient distance apart, the lower one, M_2, being near the bend at the bottom, the volume of the tube between

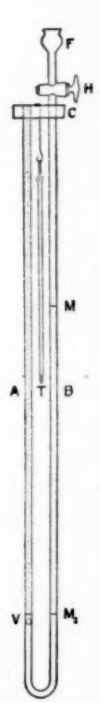

Fig. 1.
View perpendicular to the plane of the two limbs.

Fig. 2.
View in the plane of the limbs.

Fig. 3.
Steam Jacket.

the marks being accurately determined by calibration with mercury. The bore of this tube being slightly conical instead of truly cylindrical, its calibration was also carried far enough above the upper mark, M_1, to cover the space passed over by the mercury column during an experiment, so that the average height of the mercury column during an experiment could be calculated.

The method of making an observation on ether gas is essentially as follows: The steam jacket being cold and filled with air the ap-

paratus is placed in it. The stop-cock is opened and about 1 c.c. of ether is poured down the mercury tube. By means of an aspirator the ether is sucked up the other limb close to the base of the capillary. The small hermetically closed tube seen at V in Figs. 1 and 2 serves as a reservoir for ether at the base of the capillary. Care was always taken to fill this tube completely with ether so that it could not act as an air pocket. Steam is then led into the jacket and of course vaporizes the ether, driving surplus liquid violently out through the top of the mercury tube. The stop-cock is then closed and after equilibrium has been established, a weighed amount of mercury is introduced into the funnel and allowed to flow down until it is arrested by the stop-cock, which is not quite air tight, because no grease can be used in it for fear of soiling the mercury. The stop-cock is then suddenly turned and the mercury descends in a single column. When its lower meniscus passes the upper mark on the tube a stop watch is started, which is stopped as the same meniscus passes the lower mark. The barometer is read when the mercury has covered half the measured distance. A thermometer, T, Fig. 1, hanging from the cork at the top of the steam jacket indicates the temperature of the steam, and serves as a plumb line. The readings of this thermometer were corrected by comparison with a standard thermometer. The coefficient of friction of the gas is calculated by the following formula given by O. E. Meyer:[1]

$$\eta = \frac{\pi g d r^4}{16 L V} \frac{P^2 - p^2}{P} t,$$

in which d = density of mercury at 0° C.
g = acceleration due to gravitation.
r = radius of capillary in cm.
L = length of capillary in cm.
P = pressure of gas on entering the capillary.
p = height of barometer at 0° C.
t = time in seconds.
V = c.c. of gas transpired.

No allowance need be made for the expansion of the mercury at 100° C., for what the column gains in height is compensated for by

[1] Pogg. Ann., 1866, Bd. 127, p. 269.

loss in density. No allowance has been made for the expansion of the capillary as it was assumed the expansion of the mercury tube, hence increase of the volume of gas transpired, would counteract this.

The advantages of this apparatus over that devised by L. Meyer are quite obvious and numerous. In the first place it is much more simple and available. The capillary is straight instead of coiled into a helix. The friction of the vapors is taken at a temperature so high above the boiling point that they behave more like true gases than at their boiling points, where they are in a condition of unstable equilibrium. The length of time required for an experiment is short enough for all conditions to be kept constant, and yet long enough, so that with a stop watch reading to one fifth of a second, the time can be determined to within less than one part in a thousand.

Apparatus No. 1.

The first piece of apparatus was constructed more to test the practicability of the method than for getting accurate results. It having been determined by experiment, that the largest diameter of tube in which a mercury column would hold together, against a cushion of air, was about .35 cm., this apparatus was constructed with a mercury tube whose mean diameter, where it was traversed by the mercury column was .35104 cm., as shown by the fact that 76.0405 grams of mercury occupied 58.05 cm. at 20° C. The distance between the two etched marks on the tube was 50 cm. This distance was occupied by 65.5558 grams of mercury at 20° C. The density of mercury at this temperature being 13.5463 the volume between the two marks was 4.8394 cubic centimeters.

The capillary was 34.35 cm. in length which was only about half the length of the mercury tube forming the other side of the U, in other words the capillary formed the upper half of one side of the U. Its bore was conical as shown by the fact that 9.50 cm. of mercury at the small end occupied 9.05 cm. at the larger end which was made the bottom because O. Reynolds[1] has shown that converging walls tend to a steady instead of a whirling flow of water and presumably also of gas. The above column of mercury was

[1] Proc. Royal Inst. Grt. Brit., 1884, vol. 11, p. 44.

passed slowly from one end of the capillary to the other and changed its length gradually and steadily, thus showing the absence of abnormal contractions or expansions of the bore. The bore of the capillary was determined from a sample 5.75 cm. long which had been cut from the smaller end. This sample was weighed several times empty and when containing different columns of mercury and the radius of the bore determined in the well-known way. A section was also examined under a microscope with a micrometer eye piece, and found to be so nearly a true circle that it was taken for such. The microscope reading agreed well with the mercury determinations of the radius. After taking the average of the mercury and microscope readings and after allowing for the taper of the capillary its mean radius was found to be .0103908 cm.

TABLE OF THE INTERNAL FRICTION OF AIR.

Apparatus No. 1.

Temp. in C.°	Ht. of Driving Col. in Cm.	Barom. at 0° C.	Press. of Entering Gas.	Time in Seconds.	$\eta \times 10^7$.	Average $\eta \times 10^7$.
15	7.97	76.03	83.09	76.04	1,872	
14.9	7.97	76.03	83.09	75.0	1,862	
14.9	7.97	76.03	83.09	74.8	1,857	1,863.8 ± 2.57
14.9	7.966	76.03	83.09	75.2	1,866	
14.9	7.966	76.60	83.64	75.2	1,860	
15.4	7.966	76.60	83.64	75.4	1,865	
20.4	4.29	75.38	78.87	156.8	1,967	1,972 ± 3.5
20.2	4.29	75.38	78.87	157.6	1,977	
21.4	9.48	75.38	84.01	64.4	1,935	1,932 ± 2.0
21.4	9.48	75.38	84.01	64.2	1,929	
99.9	9.48	75.38	84.01	75.6	2,272.5	
99.9	9.48	75.38	84.01	76.0	2,284.5	
100.3	7.98	76.96	84.00	92.0	2,238.2	
100.3	7.98	76.96	84.00	89.8	2,185.0	
100.3	7.97	76.60	83.64	94.0	2,305.0	2,296.5 ± 15.8
100.3	7.97	76.60	83.64	96.4	2,384.5	
100.3	7.97	76.60	83.64	92.7	2,293.0	
100.3	7.97	76.60	83.64	94.0	2,325.0	

To facilitate the washing of the apparatus the small vent, *V*, in Figs. 1 and 2, was left about ½ cm. below the base of the capillary. This vent tube, which was only 1 cm. long, was hermetically

sealed off after the apparatus was washed and ready for use. The apparatus, after it was received from the glass-blower, was washed throughout with a solution of potassium permanganate made alkaline with caustic potash, then with a solution of the same substance made acid with dilute sulphuric acid. Later a half and half solution of potassium bichromate and strong sulphuric acid was substituted for the above. Sometimes also strong nitric acid, followed by dilute nitric acid, was used. Then followed many washings with distilled water. The apparatus was then dried, by sucking through it by means of an aspirator, hot air which was filtered through cotton and dried by calcium chloride. A column of calcium chloride was also placed in the connection between the aspirator and the apparatus. The drying was not hastened by the use of alcohol or ether. The mercury was cleansed by shaking with dilute nitric acid, passing through a fine pin hole and drying in a porcelain dish at 110° C.

The results of experiments on air with this apparatus are given in the preceding table. It will be noticed they are all rather high, but prove at any rate that the method and apparatus are practicable. The probable errors were calculated by the usual formula

$$0.6745 \sqrt{\frac{\Sigma' \text{diff}^2}{n(n-1)}}.$$

The error for readings at 100° C. is large, probably owing to an error of the stop-watch, whose hand showed a tendency to fly forward when stopping near 90 seconds. In one case the hand flew forward 10 seconds. The watch was of course repaired as soon as this defect was noticed. A different and better stop watch was used with apparatus No. 2 and No. 3. In those cases where the mercury was allowed to run back and was used over again the readings were weighted less than independent readings, three readings being converted into two by taking their means, and four into three in the same way.

It is rather difficult to tell what is the true coefficient of the internal friction of air. Landolt and Boernstein[1] give η for air at 15° C. $1{,}784 \times 10^7$. Markowski[2] at 16° gives $1{,}814 \times 10^7$, Kleint[3]

[1] Phys. Chem. Tabellen, Landolt and Boernstein, 1893.
[2] Inaug. Diss. Halle, 1903.
[3] Inaug. Diss. Halle, 1904.

at 14.1 to 14.5° C. gives $1{,}808 \times 10^7$. According to Landolt and Boernstein, then, my readings are 4½ per cent. too high at 15°, while according to Markowski and Kleint they are about 3 per cent. too high. F. G. Reynolds[1] gives for air at 20.7° .000187, which is 3¼ per cent. lower than my result of .0001932 at 21.4° C.

It was found by repeated trials with di-ethyl ether that the surface tension of the mercury was so much reduced by contact with the ether gas that the mercury would not hold together in one column in apparatus No. 1. It was determined by experiment that the mercury piston could not be used for ether gas in a tube whose diameter was much larger than 2 mm. A second piece of apparatus was accordingly constructed with a mercury tube of about this size, and a finer capillary, so that the transpiration time would be increased and greater accuracy be secured.

APPARATUS NO. 2.

The capillary selected for this when examined with a simple microscope was at first thought to be circular in cross-section, but the use of a high-power microscope with micrometer eye-piece showed it to be very elliptical, the ratio of the axes being almost exactly as 3 is to 1. The area of its cross-section was determined by mercury several times, and these values averaged with the microscope reading with which they agreed well.

The capillary which was 84 cm. long tapered in its bore, 2.5 cm. of mercury at the large end becoming 2.685 cm. at the small end. The sample whose bore was determined was taken from the small end. After allowing for the taper of the bore the average semi-major axis was found to be .006057 cm. and the average semi-minor axis .002016 cm. The capillary was placed with its larger end downward and reached to within 15 cm. of the bend in the U.

The mean diameter of the mercury tube in that part traversed by the mercury piston during an experiment was .2012 cm. as shown by the fact that 31.3 grams of mercury occupied a length of 72.6 cm. at 16° C. The marks on the mercury tube were made 50 cm. apart, the volume between these marks being found by means of mercury to be 1.5707 c.c.

[1] PHYS. REV., 1904, vol. 18, p. 419; vol. 19, p. 37.

Because the capillary was elliptical the formula used for calculating the internal friction of gas from this apparatus is, according to Mathieu,[1]

$$\eta = \frac{\pi dg}{8LV} \frac{a^3b^3}{a^2+b^2} \frac{P^2-p^2}{P} t$$

where a = semi-major axis of ellipse of capillary.
b = semi-minor axis of ellipse of capillary.
d = density of mercury at 0° C..
g = acceleration due to gravitation.
L = length of capillary.
V = vol. of gas transpired.
P = pressure of entering gas.
p = pressure of leaving gas = barom. at 0° C.
t = time in seconds.

The results with air with this apparatus are shown in the next table.

It will be noticed that the values of η are somewhat lower than for apparatus No. 1, and therefore nearer the correct values.

The different lengths of the capillary given in the first column are due to the fact that on several occasions the upper end of the capillary became stopped with dust from the atmosphere and had to be cut off. When not in use the capillaries were kept capped with rubber, the funnels filled with cotton and closed by corks in which were inserted tubes of chloride of calcium.

TABLE OF THE INTERNAL FRICTION OF AIR.

Apparatus No. 2.

Length of Cap.	Temp. in C.°.	Ht. of Driving Col.	Barom. at 0° C.	Press. of Entering Gas.	Time in Seconds.	$\eta \times 10^7$.	Average $\eta \times 10^7$.
81.8	12.5	23.17	75.74	97.92	2,604	1,864.2	
84	18.5	23.17	76.19	98.36	2,721	1,897.4	
84	19.9	23.15	76.26	98.41	2,715	1,892.0	1,897.3 ± 3.5
84	19.9	23.42	76.41	98.82	2,700	1,902.5	
84	20.9	23.15	76.26	98.41	2,730	1,902.2	
84	26.9	23.16	76.15	98.31	2,810	1,957.1	
84	29.1	23.17	75.90	98.08	2,830	1,973.6	
84	100	23.17	75.91	98.09	3,201	2,233.0	
81.8	99.9	23.14	75.70	98.85	3,130	2,237.0	2,242 ± 5
81.8	100.2	23.14	76.51	98.65	3,155	2,257.0	

[1] Compt. Rend., 1863, Tome 57, p. 320.

Apparatus No. 3.

73.25	10	18.746	76.78	94.56	509.8	1,843.8	
73.25	13.3	18.75	76.78	94.56	516	1,866.9	
73.25	14.4	18.74	76.80	94.58	518.2	1,870.0	
73.25	19.4	18.75	76.00	93.79	524.6	1,894.0	
73.25	20.8	18.74	75.99	93.78	531.2	1,917.2	
70	21.0	18.75	76.99	94.77	509.2	1,923.6	1,909.9 ± 7.2
70	20.8	18.75	76.95	94.74	500.0	1,889.0	
70	100	18.74	75.76	94.50	592.8	2,244.0	
70	100.1	18.74	76.19	94.93	599	2,267.0	2,246.3 ± 7.2
70	100.3	18.75	77.13	94.92	588.8	2,228.0	

PREPARATION OF ETHERS.

The di-ethyl ether used in the following experiments was sulphuric ether, U. S. P. purified by two washings with concentrated sulphuric acid, C. P., each of these washings being followed by one with distilled water. It was then shaken with mercury and dried with sodium wire from which it was distilled into glass tubes which were afterward hermetically sealed. The ethyl-propyl ether, methyl-propyl-ether and methyl-ethyl ether used in this investigation were prepared by me, in the research laboratory of New York University under the personal supervision of Professor Loeb, according to the continuous etherification method described by Norton and Prescott in the Am. Chem. Journal, 1884, Vol. VI., p. 241. They were carefully dried with sodium and kept in sealed tubes.

The other substances, viz.: di-methyl ether, ethyl alcohol, methyl-isopropyl ether, ethyl-isopropyl ether, di-propyl ether, isopropyl-propyl ether and di-isopropyl, I owe entirely to the great courtesy of Professor Loeb, as I had no part at all in their preparation. As far as is known this is the first time that methyl-isopropyl ether has ever been made, while ethyl-isopropyl ether and isopropyl-propyl ether have probably been made only once before.

The internal frictions of the ethers used in apparatus No. 2 are given in the foregoing table on page 239.

The ethers are arranged in this table in the order of their molecular weights; that of methyl-ethyl ether being 60.064, that of methyl-propyl, methyl-isopropyl, di-ethyl ether being alike 74.08; while that of ethyl propyl is 88.096. It will be noticed the smaller the molecular weight the greater is the internal friction, which

Table of Internal Friction of Ether Gases.

Apparatus No. 2.

Length Capillary.	Kind of Gas.	Temp. of Gas.	Ht. of Driving Col. at 0° C.	Barom. at 0° C.	Press. of Entering Gas.	Time in Seconds.	$\eta \times 10^7$.	Average $\eta \times 10^7$.
81.8	Methyl-	99.9	23.16	75.68	97.85	1,502.0	1,074.5	
81.8	Ethyl	100	23.20	76.09	98.30	1,537.8	1,103	
81.8	Ether.	99.9	23.19	75.87	98.07	1,471.0	1,053.9	
81.8		99.3	23.16	75.44	97.61	1,558.0	1,114.2	1,092.3 ± 6.4
81.8		100	23.06	76.28	98.35	1,550.2	1,105.4	
81.8		100	23.16	76.15	98.32	1,540.0	1,102.8	
84	Methyl-	100	23.90	76.28	99.18	1,380.0	990.9	
84	Propyl	100	23.22	76.23	98.46	1,409.6	985.4	
84	Ether.	100	23.28	76.18	98.47	1,402.2	982.5	
84		100.1	23.14	76.46	98.60	1,461.8	1,018.9	1,004.3 ± 5.5
82.5		99.9	23.13	75.31	97.46	1,440.4	1,021.2	
81.8		99.8	23.11	74.24	96.39	1,439.9	1,027.0	
81.8	Methyl-	100.3	23.15	77.15	99.30	1,481.4	1,061.1	
81.8	Isopropyl	100.3	23.14	77.10	99.24	1,526.0	1,081.1	
81.8	Ether.	100	23.13	76.01	98.24	1,459.6	1,047.3	1,054.8 ± 5.0
81.8		100	23.12	76.00	98.13	1,455 0	1,039.9	
81.8		100	23.14	76.00	98.15	1,460.8	1,044.9	
84	Di-Ethyl	100.1	23.42	76.39	98.81	1,410.0	993.4	
84	Ether.	100	23.42	76.08	98.51	1,404.0	990.6	
84		100	23.16	76.15	98.32	1,416.6	986.8	
84		100	23.17	76.10	98.28	1,417.6	988.6	999 ± 3.3
84		100.1	23.16	76.54	98.71	1,437.6	1,000.6	
82.5		99.9	23.16	75.71	97.89	1,434.6	1,017.8	
82.2		99.9	23.21	75.50	97.73	1,428.0	1,015.2	
84	Ethyl-	100	23.17	76.06	98.24	1,285.0	896.2	
84	Propyl	100	23.17	75.89	98.07	1,296.4	904.6	
84	Ether.	99.9	23.16	75.70	97.87	1,281.2	892.7	
81.5		99.9	23.15	75.58	97.75	1,295.8	930.5	915.2 ± 5.3
81.5		99.9	23.14	75.57	97.73	1,298.8	932.3	
81.5		99.9	23.15	75.57	97.74	1,300.8	934.0	

agrees well with the kinetic theory of gases, according to which the friction increases with diminished size of molecule.

The most noteworthy fact shown in this table is that the three isomeric ethers, di-ethyl, methyl-propyl and methyl-isopropyl have not the same internal friction. Di-ethyl ether and methyl-propyl practically agree, while methyl-isopropyl ether has a friction 5 per cent. higher. According to the generally accepted view of the

concatenation of the molecules of these substances, the first two are simple chain compounds, while the last is a chain compound with two branches:

Di-ethyl ether $H_3C—C(H_2)—O—C(H_2)—CH_3$

Methyl-propyl ether $H_3C—O—C(H_2)—C(H_2)—CH_3$

Methyl-isopropyl ether $H_3C—O—C(H)(CH_3)(CH_3)$

It is easily conceivable how the last arrangement results in a compacter, smaller molecule, and hence shows a higher internal friction.

Although the amount of liquid used varied from ½ c.c. to 1 c.c. as a rule, this small amount seemed sufficient on vaporizing to drive all the air out of the apparatus, as filling the apparatus entirely with ether from the base of the capillary to the stop-cock did not give results different from those obtained using 1 c.c. of ether.

The weight of mercury used in apparatus No. 1 and No. 2 was usually about 10 grams. That in apparatus No. 3 shortly to be described was about 5 grams.

The allowance to be made for the friction of the mercury against the walls of the tube was determined in the following manner. Three readings with air were taken with three very different columns of mercury, under constant temperature conditions. It being known that the internal friction of a gas is independent of the pressure, the values of η from these three readings were equated to each other in pairs, giving, after cancelling out constants of the apparatus, equations of the form

$$\frac{P^2c^2 - p^2}{Pc}t = \frac{P_1^2c^2 - p_1^2}{P_1c}t_1,$$

in which c is a constant reduction factor due to the friction of the mercury against the walls of the tube. The effect of capillarity is probably negligible, because the upper and lower surfaces of the mercury are convex in opposite directions.

In solving the equations for c good agreement was found, the

average c for apparatus No. 1 being .9892. For apparatus No. 2 $c = .9895$ and for apparatus No. 3 $c = .9894$, all at 100° C. The pressure of the gas on entering the capillary was calculated by adding the height of the mercury column at 0° C. to the barometer at 0° C. and multiplying by the reduction factor given above. The mercury and barometer column were reduced to 0° C. by the aid of a table given on page 248 of Kohlrausch's Kleiner Leitfaden der Praktischen Physik.

Apparatus No. 3.

In order to have a second piece of apparatus available for experiments with ether, so that two experiments could be made at the same time, a third piece of apparatus was constructed. The mercury tube of this apparatus was made a little smaller than that of apparatus No. 2, as some difficulty had been met in the mercury column failing to hold together properly. The capillary of this apparatus was also elliptical and slightly conical. The average semi-major axis as determined by the microscope and mercury was .006176 cm. while the minor axis was .002837. The length of the capillary was 73.25 cm. at first, but was reduced to 70 cm. when the apparatus was repaired after a breakage.

The distance between the marks on the mercury tube was 38.45 cm. and the volume of gas transpired was .742 c.c. since 10.0634 grams of mercury occupied the distance between the marks at 16° C. The average diameter of the mercury tube where the mercury traversed it during an experiment was .158112 cm.

A table of the results obtained with this apparatus for air is given on page 238 underneath those for air with apparatus No. 2. It will be noticed that the two pieces of apparatus agree very closely in results.

The results obtained for ethers with this apparatus are given in the following table. The first point worthy of note is that ethyl alcohol which is metameric with di-methyl ether has about three per cent. lower coefficient of friction, showing it to have the larger molecule.

The values for di-ethyl ether, and ethyl-propyl ether agree well with those given on page 239 by apparatus No. 2.

Ethyl-isopropyl ether has a friction 4.6 per cent. higher than that o ethyl-propyl ether, hence a smaller molecule.

Table of Internal Friction of Ether Gases.

Apparatus No. 3.

Length Capillary.	Kind of Gas.	Temp. of Gas.	Ht. of Driving Col. at 0° C.	Barom. at 0° C.	Press. of Entering Gas.	Time in Seconds.	$\eta \times 10^7$.	Average $\eta \times 10^7$.
70	Di-methyl	100.3	18.75	77.24	95.03	315.6	1,191.8	
70	Ether.	100.3	18.74	77.32	95.10	314.8	1,188.6	1,190.2 ± 1
70	Ethyl	100.3	18.74	76.98	94.67	303.6	1,145.6	
70	Alcohol.	100.3	18.76	76.99	94.70	308.4	1,164.4	1,155 ± 6.3
73.25	Di-ethyl	100.1	18.78	76.43	94.26	287.4	1,041.8	
73.25	Ether.	100	18.60	76.15	93.81	287.4	1,023.2	
73.25		100	18.79	76.03	93.88	270.4	978.4	1,001 ± 8.15
70		99.9	18.76	75.80	93.61	254.8	963.7	
70		100.1	18.56	76.46	94.07	266.5	998.1	
73.25	Ethyl-	99.9	18.76	75.79	93.61	253.2	914.8	
73.25	Propyl	99.9	18.26	75.74	93.06	267.2	939.3	
73.25	Ether.	100	18.05	76.39	93.50	262.4	921.7	918.4 ± 4.2
73.25		100.3	18.28	76.83	94.16	256.6	902.7	
73.25		100.4	18.05	77.02	94.14	263.2	913.7	
70	Ethyl	100.2	17.74	76.46	93.26	267.7	946.4	
70	Isopropyl	100.2	18.76	76.41	94.22	255.4	966.4	960.7 ± 5
70	Ether.	100.2	18.56	76.34	93.95	258.8	969.3	
70	Di-Propyl	100	18.75	75.95	93.73	222.2	839.5	
70	Ether.	100	18.76	76.02	93.74	222.0	839.3	837.5 ± 1.26
70		100.3	18.75	77.14	94.93	220.8	832.7	
70	Isopropyl-	100.1	18.73	76.19	93.97	230.4	870.2	873.4 ± 2.2
70	propyl Ether.	100.1	18.76	76.22	94.03	231.6	876.6	
70	Di-isopro-	100.3	18.74	77.14	94.92	230.2	869.2	
70	pyl	100.3	18.74	77.14	94.92	230.8	871.5	
70	Ether.	100.3	18.76	77.09	94.89	239.0	902.8	894.3 ± 5.95
70		100.3	18.74	77.13	94.91	236.0	890.7	
70		100.1	18.73	76.23	94.01	241.0	910.5	
70		100.1	18.75	76.23	94.03	243.6	921.2	

Ethyl-propyl ether $H_3C—\underset{H_2}{C}—O—\underset{H_2}{C}—\underset{H_2}{C}—CH_3$

Ethyl-isopropyl ether $H_3C—\underset{H_2}{C}—O—\underset{H}{C}<\begin{matrix}CH_3\\CH_3\end{matrix}$

Isopropyl-propyl ether has a friction 4.3 per cent. higher than that of di-propyl ether, while di-isopropyl ether has a friction 6.8

per cent. higher than di-propyl. This is in accordance with our view of the concatenation of the atoms in these molecules. Di-propyl ether is a straight, chain compound whereas isopropyl-propyl has two branches and di-isopropyl has four branches.

Di-propyl ether $H_3C—\underset{H_2}{C}—\underset{H_2}{C}—O—\underset{H_2}{C}—\underset{H_2}{C}—CH_3$

Isopropyl-propyl ether $\begin{matrix}H_3C\\H_3C\end{matrix}\!\!>\!\!C\underset{H}{}—O—\underset{H_2}{C}—\underset{H_2}{C}—\underset{H_3}{C}$

Di-isopropyl ether $\begin{matrix}H_3C\\H_3C\end{matrix}\!\!>\!\!C\underset{H}{}—O—\underset{H}{C}\!\!<\!\!\begin{matrix}CH_3\\CH_3\end{matrix}$

As already noticed the difference between methyl-propyl and methyl-isopropyl is 5 per cent. due to two branches; the difference between ethyl-propyl and ethyl-isopropyl due to two branches is 4.6 per cent.; the difference between di-propyl and isopropyl-propyl is 4.3 per cent. due to the same number of branches. The diminishing effect of the two CH_3 branches is probably due to the fact that the molecules of the substances lower on the lists are larger, hence the branches have a smaller relative effect. It would seem that di-isopropyl ether ought to be 8.6 per cent. higher than di-propyl whereas it is only 6.8 per cent. This discrepancy I am unable to explain.

Di-methyl ether, being a gas at ordinary temperatures, had to be handled differently from the other ethers, all of which were put into the apparatus as liquids. The low boiling ethers, methyl-ethyl, and methyl-isopropyl were experimented upon in a room whose temperature was at or very near 0° C.

The di-methyl ether gas was kept in sealed glass bulbs of 250 c.c. capacity. The lower end of a bulb was connected by rubber tubing to a reservoir of mercury while the upper end was connected by a rubber tube to the funnel at the top of the apparatus. After the aspirator had created a partial vacuum in the apparatus by sucking the air out through the capillary, the tips of the bulb were broken off inside the rubber tubes and mercury allowed to flow into the bulb gradually from the bottom, driving the ether gas before it into the apparatus, the action being assisted all the while by the suction

of the aspirator. As 250 c.c. was many times the cubic capacity of the apparatus, by the time the mercury had entirely filled the ether bulb, it was judged the apparatus would be filled with pure ether, unmixed with any air. This method of getting the ether gas into the apparatus proved a failure several times for various reasons, but the very close agreement of the two readings which were at length obtained points to their correctness.

Molecular Volumes.

L. Meyer[1] gives the following formula for calculating approximate relative molecular volumes, derived from the molecular volume of SO_2 as determined by Andreef at — 8° C.

$$V = 3.10^{-6}\{M(1 + \alpha t)\}^{\frac{3}{4}}\left\{\frac{1}{\eta}\right\}^{\frac{3}{2}}$$

in which M = molecular weight, t = temperature Centigrade η = coefficient of friction.

The values obtained for the substances under investigation are given on the next page. According to Kopp[2] the molecular volume of a liquid composed of carbon, hydrogen and oxygen can be found by substituting in its formula 11 for each atom of carbon, 5.5 for each hydrogen and 7.8 for each oxygen. The values thus calculated are given in the last column. The agreement between the molecular volumes calculated in these two ways is quite good, much better than that obtained by L. Meyer and Schumann.

As both apparatus No. 2 and No. 3 gave values for air about 5 per cent. higher than those given by Landolt and Boernstein as probably the most correct values, it is probable that the values of η for the ethers are correspondingly too high. Accordingly in the second table of molecular volumes the values of η were reduced by multiplying by 2,113/2,244, thus calibrating with air, and the molecular volumes recalculated. A still better agreement with Kopp's values is shown, the disagreement being greatest when the boiling point of the ether approaches the temperature at which η was determined.

[1] Wied. Ann., 1881, Bd. 13, p. 17.
[2] Ann. Chem. Phar., 1855, Bd. 96, pp. 1, 153, 303.

First Table of Molecular Volumes.

Material.	Formula.	Boiling Point in Degrees Centigrade.	$\eta \times 10^7$.	Molec. V. Found.	Molec. V. Calculated According to Kopp.
Di-methyl ether.	C_2H_6O	—23.65	1,190.2	51.62	62.8
Ethyl alcohol.	C_2H_6O	78.	1,155.0	55.25	
Methyl-ethyl ether.	C_3H_8O	10. –13.	1,092.3	71 65	84.8
Di-ethyl ether.	$C_4H_{10}O$	34.6–35.	1,000	95.73	106.8
Methyl-propyl ether.	$C_4H_{10}O$	40. –45.	1,004.3	95.16	
Methyl-isopropyl ether.	$C_4H_{10}O$	7.	1,054.8	88.37	
Ethyl-propyl ether.	$C_5H_{12}O$	66. –68.	916.8	124.19	128.8
Ethyl-isopropyl ether.	$C_5H_{12}O$	54. –57.	960.7	115.77	
Di-propyl ether.	$C_6H_{14}O$	84.5–86.5	837.5	158.90	150.8
Isopropyl-propyl ether.	$C_6H_{14}O$	76.5–77.	873.4	148.86	
Di-isopropyl ether.	$C_6H_{14}O$	68.5–69.	894.3	144.06	

Second Table of Molecular Volumes.

Material.	Formula.	Boiling Point in Degrees Centigrade.	$\eta \times 10^7$.	Molec. V. Found.	Molec. V. Calculated According to Kopp.
Di-methyl ether.	C_2H_6O	—23.65	1,121.0	56.46	62.8
Ethyl alcohol.	C_2H_6O	78.	1,088.0	59.05	
Methyl-ethyl ether.	C_3H_8O	10. –13.	1,029.0	81.12	84.8
Di-ethyl ether.	$C_4H_{10}O$	36.6–35.	942.0	104.71	106.8
Methyl-propyl ether.	$C_4H_{10}O$	40. –45.	946.0	104.20	
Methyl-isopropyl ether.	$C_4H_{10}O$	7.	993.6	96.66	
Ethyl-propyl ether.	$C_5H_{12}O$	66. –68.	863.6	135.84	128.8
Ethyl-isopropyl ether.	$C_5H_{12}O$	54. –57.	905.1	126.61	
Di-propyl ether.	$C_6H_{14}O$	84.5–86.5	788.9	173.08	150.8
Isopropyl-propyl ether.	$C_6H_{14}O$	76.5–77.	822.7	163.20	
Di-isopropyl ether.	$C_6H_{14}O$	68.5–69.	842.4	157.51	

In this second table η has been reduced by calibration with air.

Molecular Mean Speeds, Free Paths and Collision Frequencies.

In Meyer's Kinetic Theory of Gases on page 219 is given the formula $\eta = 0.30967\ \rho L \Omega$ whence

$$L = \frac{\eta}{0.30967 \rho \Omega},$$

in which η = coefficient of internal friction of the gas,

ρ = density of the gas,

Ω = mean molecular velocity,

L = mean molecular free path.

On page 55 of the same work we find $p = \frac{1}{8}\pi\rho\Omega^2$ whence

$$\Omega = \sqrt{\frac{8p}{\pi\rho}},$$

where p = pressure in absolute measure.

On page 195 we see that the

$$\text{Collision Frequency} = \frac{\Omega}{L}.$$

The values given in the table on this page were calculated according to the above formulæ, at a pressure of 76 cm. of mercury and a temperature of 100° C., the density being calculated from the formula

$$\rho = \tfrac{1}{2}\frac{(\text{formula weight})\ (0.000089)}{1 + at}.$$

First Table of Mean Molecular Speeds, Free Paths and Collision Frequencies.

Material.	Molecular Weight.	$\eta \times 10^7$.	Mean Molecular Speed Cm. per Sec.	Molecular Free Path Cm. $\times 10^{10}$.	Collision Frequency $\times 10^{-6}$.
Di-methyl ether.	46.048	1,190.2	41,470	61,787	6,711.7
Ethyl alcohol.	46.048	1,155.0	41,470	59,970	6,915.0
Methyl-ethyl ether.	60.064	1,092.3	36,310	49,660	7,312.0
Di-ethyl ether.	74.08	1,000.0	32,695	40,895	7,994.9
Methyl-propyl ether.	74.08	1,004.3	32,695	41,111	7,952.7
Methyl-isopropyl ether.	74.08	1,054.8	32,695	43,180	7,572.0
Ethyl-propyl ether.	88.096	916.8	29,982	34,415	8,711.8
Ethyl-isopropyl ether.	88.096	960.7	29,982	36,063	8,315.5
Di-propyl-propyl ether.	102.112	837.5	27,848	29,201	9,536.6
Isopropyl-propyl ether.	102.112	873.4	27,848	30,454	9,144.6
Di-isopropyl ether.	102.112	894.3	27,848	31,182	8,931.0

Second Table of Mean Molecular Speeds, Free Paths and Collision Frequencies.

Di-methyl ether.	46.048	1,121.0	41,470	58,204	7,125.0
Ethyl alcohol.	46.048	1,088.0	41,470	56,490	7,341.0
Methyl-ethyl ether.	60.064	1,029.0	36,310	46,782	7,762.0
Di-ethyl ether.	74.08	942.0	32,695	38,570	8,477.0
Methyl-propyl ether.	74.08	946.0	32,695	38,725	8,639.5
Methyl-isopropyl ether.	74.08	993.6	32,695	40,682	8,036.7
Ethyl-propyl ether.	88.096	863.6	29,982	32,418	9,248.4
Ethyl-isopropyl ether.	88.096	905.1	29,982	33,976	8,824.3
Di-propyl-propyl ether.	102.112	788.9	27,848	27,507	10,124.0
Isopropyl-propyl ether.	102.112	822.7	27,848	28,685	9,708.0
Di-isopropyl ether.	102.112	842.4	27,848	29,372	9,481.1

In this second table η has been reduced by calibration with air.

It will be noticed that the values are all of the proper order of magnitude, but do not rise and fall in a periodic way while the molecular weight increases, as pointed out for other substances by Meyer on page 196 of his Kinetic Theory of Gases, probably because these ethers are all so closely related.

Comparison with the Results of Others.

The results of my work cannot be compared satisfactorily and exactly with that of others, because I have determined the friction at a temperature higher than most other observers, and the law of its increase with the temperature is not accurately known in each case. Furthermore, air, di-ethyl ether, di-methyl ether and ethyl alcohol are the only substances which I have employed that others have investigated.

Disregarding the results with apparatus No. 1 as being only preliminary, we see that the values of η for air for the other two pieces of apparatus agree closely and are consistent, though both are about 5 per cent. higher than the values which are most probably correct, according to Landolt and Boernstein's tables. That the relative values of η at the different temperatures are correct is shown by calculating η at 0° C. by Sutherland's formula[1] and then calculating from that η at 100° C. Using the value of the cohesion constant $C = 119.4$ as determined by Breitenbach[2] we get for apparatus No. 2, $\eta_0 = .0001793$ and $\eta_{100} = .00022367$ the latter value agreeing with .0002242 observed within $\frac{1}{4}$ per cent. For apparatus No. 3 we get in like manner $\eta_0 = .0001795$ and $\eta_{100} = .0002239$, the latter value agreeing with .0002246 observed within $\frac{1}{3}$ per cent. This agreement is within the limits of error of the experiments.

Puluj's[3] formula for the coefficient of friction of di-ethyl ether vapor, $\eta = 0.0000689\ (1 + .0041575\, t)^{.94}$ gives $\eta = 0.0000935$ at 100° C. which agrees with my corrected value of 0.0000942 within $\frac{3}{4}$ per cent. This is good agreement bearing in mind the fact that Puluj's formula was determined from experiments over a small range of temperature, viz.: from 0° to 37° C.

[1] Phil. Mag., 1893 (5), vol. 36, p. 507.

[2] Drude's Ann., 1901, Bd. 5, p. 166.

[3] Wien. Ber., 1878, vol. 78 (2), p. 279; Carls Rep., 1878, vol. 14, p. 573; Phil. Mag., 1878 (5), vol. 6, p. 157.

The coefficient of friction of di-methyl ether is given on page 192 of Meyer's Kinetic Theory of Gases as 0.000092 at 0° C. No one has determined the law of its increase with the temperature. My corrected value is .0001121 at 100° C. which seems reasonable.

Steudel [1] gives 0.000142 as the coefficient of friction of ethyl alcohol at 78.4° C., its boiling point. My corrected value of .0001088 at 100° C. does not agree well with this. Because his value was determined at the boiling point I think it is open to question. My value for ethyl alcohol agrees much better with Puluj's [1] results which are 0.0000827 at 0° C. and 0.0000885 at 16.8° C. He assumes η is proportional to the absolute temperature which gives $\eta = .0001130$ at 100° C., which is nearly 4 per cent. higher than my value of .0001088. I think the assumption that η increases exactly as the absolute temperature is not strictly correct.

Concerning the differences which I have found between the normal propyl and the isopropyl ethers I would point out that Lothar Meyer, Schumann and Steudel found similar differences between many, though not all, of the normal propyl and isopropyl compounds which they examined. They also found that normal butyl, isobutyl and tertiary butyl compounds showed still more regular differences. The weight of evidence gives a larger molecule to the primary, a smaller to the secondary and the smallest to the tertiary compound. My values of the molecular speeds, free paths and collision frequencies being for 100° C. of course do not agree with those calculated for 0° C. by others.

SUMMARY OF RESULTS.

1. The coefficients of internal friction of the following eight ether gases which have hitherto not been experimented with, have been determined with considerable accuracy as follows:

Methyl-ethyl ether	0.0001029	at 100° C.
Methyl-propyl ether	0.0000946	" "
Methyl-isopropyl ether	0.00009936	" "
Ethyl-propyl ether	0.00008636	" "
Ethyl-isopropyl ether	0.00009051	" "
Di-propyl ether	0.00007889	" "
Iso-propyl-propyl ether	0.00008227	" "
Di-isopropyl	0.00008424	" "

[1] Wied. Ann., 1882, Bd. 16, p. 369.

2. The molecular volumes calculated from the friction have been shown to agree fairly well with those obtained by Kopp's rule.

3. A marked and unmistakable difference between the normal propyl and isopropyl ethers has been found, proving that the difference in the molecular structure of these ethers has a very noticeable effect upon their internal friction, and therefore upon the size of their molecules, the molecules having the most numerous branches being smaller than those with fewer or no branches.

In conclusion I wish to express my thanks to Professor Loeb for suggesting both the subject of this research and the form of apparatus, and for his kind interest and help throughout the course of the investigation. In addition I wish to thank him heartily for his courtesy in supplying me with certain of the necessary ethers.

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THE DISTRIBUTION OF ENERGY EMITTED BY A RIGHI VIBRATOR.

By C. R. Fountain and F. C. Blake.

Introduction.

HERTZ[1] found that most of the energy emitted by his oscillator was radiated in the equatorial plane, that is, the plane through the spark gap and perpendicular to the axis of the oscillator; and that little or no energy was radiated in the polar direction. He mapped the direction of the electric force about the oscillator, but his results were qualitative rather than quantitative on account of the type of detector used.

In developing the theoretical side Hertz[2] failed to take into account the damping of the waves, which V. Bjerkness,[3] Hull[4] and others have shown to be large. Karl Pearson and Alice Lee[5] have shown how this factor would affect the drawings which Hertz made to represent the field about the oscillator. But they neglected the fact that there is an electrostatic field about the oscillator before the oscillation begins. A. E. H. Love[6] has considered theoretically all these factors and shown how the electric and magnetic forces may be computed for any point in the field.

It is the purpose of the present paper to show the relative amounts of energy radiated in the various directions about a Righi vibrator, and to indicate what bearing the experiments to be described have upon existing theories.

Apparatus.

The general arrangement of the apparatus is shown in Fig. 1. The vibrator V is placed at the center of the circle H. The distance

[1] Electric Waves, Eng. ed., p. 87.
[2] Electric Waves, Eng. ed., p. 141.
[3] An. der Phys., Vol. 44, pp. 81 and 513.
[4] Phys. Rev., Vol. 5, p. 231, 1897.
[5] Phil. Trans. Royal Soc. of Lon., Ser. A, V. 193, p. 158.
[6] Proc. Roy. Soc., Vol. 74, p. 73, 1904.

r of the receiver R is varied by sliding along a meter bar MM'. The angle which the radius vector at any point makes with the equatorial (vertical) plane could be varied by swinging the circle H about the axis NN' and its position at any time could be read on a graduated disk DD'. This angle will hereafter be denoted by θ.

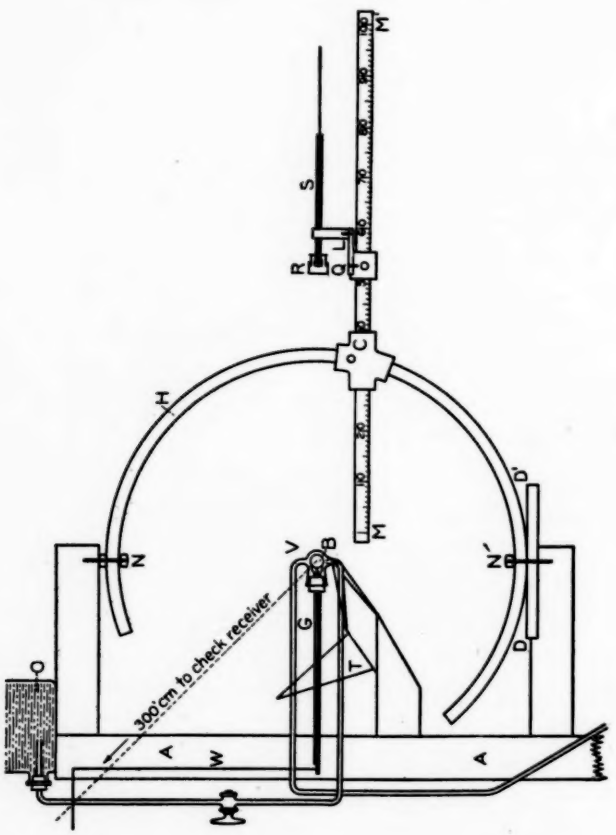

Fig. 1.

The vertical angle could be changed by sliding the clamp C about the graduated circle H. At any point the orientation of the receiver R could be changed by rotating the rod S about its axis and also by rotating the support arm L about the pivot Q. All the parts shown are drawn to scale and can be compared with the meter bar MM'. The large bottle of oil, O, on top of the support AA' supplied a constant stream of olive oil between the vibrator balls BB'

to carry off any carbon particles formed by the electrical decomposition of the oil. The brass balls BB', 2.54 cm. in diameter, were sealed with ordinary red sealing wax to the enlarged ends of the glass tubes G. These were fitted into the glass vessel V. The wires W from a 10-inch induction coil passed through the tubes G and terminated in small beads about .3 cm. from the balls B. This permitted the balls to be charged by small sparks through air. The

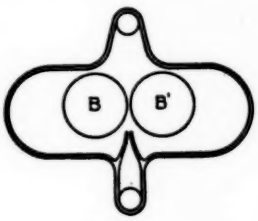

Fig. 2.

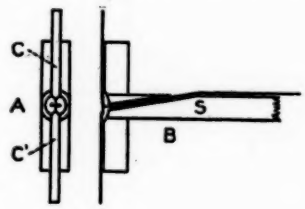

Fig. 3.

oscillation between the balls BB' took place through a small film of olive oil. An enlarged view of a vertical cross-section through the axis of the vibrator is shown in Fig. 2. This view shows the relative positions of the balls in the glass vessel. The tubes G led directly back so as to affect as little as possible the field in front of the vibrator. The triangular block T was placed to prevent interference with waves reflected from the post AA'. No metallic nails were used in constructing the apparatus.

Receiver.

The wave-detector was the type of thermo-element used in a former paper,[1] differing mainly in dimensions, but the contact between the iron and constantan wires was electrically welded instead of soldered. Fig. 3 shows two views of the receiver and R, Fig. 1, gives an end view showing the cardboard box surrounding it. The iron and constantan wires each .0025 cm. in diameter were soldered to the copper strips CC' (Fig. 3, A) crossed in the center and soldered to small copper wires which were drawn backward through the rod S (Fig. 3, B) so as to leave as little as possible (about .15 cm.) of the wires projecting at right angles to the axis of the copper strips. The total length of the receiver was 8 cm.

[1] Blake and Fountain, PHYS. REV., Vol. XXIII., No. 4, p. 257, 1906.

The strips CC' were .3 cm. wide and the length of wire between them was .35 cm.

Since a vibrator does not always give off the same amount of energy a check receiver was employed and the relative amounts of energy denoted by the ratios M/C, where M denotes the scale reading for the galvanometer connected to the main receiver and C the corresponding reading for the check receiver. To keep the two receivers from affecting each other the check receiver was placed on a shelf 3 meters distant near the focus of a spherical metallic mirror of 25 cm. diameter and 16 cm. focal length.

The same induction coil and galvanometers used in an earlier experiment[1] served in the present study.

TESTS FOR DISTURBING FACTORS.

Before beginning the exploration of the field it was necessary to make sure that no disturbing factors were neglected. A few of the more important tests are here given:

1. When the two spheres B, B' were made to touch each other and the induction coil put in operation, no effect could be detected in either galvanometer unless one of the receivers was within 10 cm. of a lead wire W.

2. Notwithstanding this a further test for effects from the induction coil was made by taking readings with the coil in various orientations at its regular distance of 400 cm. from the vibrator and 900 cm. from the galvanometers. The greatest variation in the ratios M/C, even when the main receiver was at the pole so as to receive but little of the energy radiated from the vibrator, was less than 5 per cent. The coil was then placed permanently in the position to give least effect.

3. The vibrator axis was set obliquely to the walls of the room so as to avoid all chance of stationary waves and disturbing reflections. Later tests showed the complete absence of any energy reflected from surrounding objects except a small amount reflected from the post AA', Fig. 1. The triangular block T was inserted to divert these reflections.

4. The ratio M/C should be constant for all points equidistant in

[1] Blake and Fountain, l. c.

the equatorial plane. This was true for all angles from 45° below the horizontal to 85° above, except in the neighborhood of 45° above when an increase of nearly 4 per cent. was observed for a distance of 20 cm. For this region however it was impossible to cut off the reflections from the post AA' without interfering with the waves en route to the check receiver.

5. One of the most troublesome disturbances and one not discovered for some time was the effect of one receiver upon the other. Further experiments showed that a copper strip, the same length as a receiver, when placed in the vicinity of the vibrator might increase or decrease the ratio M/C by 50 per cent. In other words the receivers reradiate most of the energy they receive. It was for this reason that the check receiver was placed so far away near the focus of a spherical mirror. It was slightly displaced from the focus so that the energy reradiated from it would be brought to a focus somewhere above the horizontal plane. If the main receiver were 5 cm. from the vibrator the energy reradiated from it would be brought to focus sufficiently near the check receiver to nearly double its readings. If placed 10 cm. away, however, the effect could scarcely be detected. But to make sure that no effect of this kind should enter the final readings no attempt has been made to explore the field within a distance of 14.4 cm.

6. The galvanometer wires leading to the thermal junction were closely twisted to within about .7 cm. from the thermal junction. But if these twisted wires formed a loop where the field was strong and projected in the direction of the electric force it might absorb nearly as much energy as the receiver itself. That is, if the receiver axis were perpendicular to the electric force and a loop of about 3 cm. diameter were parallel to the electric force, the galvanometer deflection might be nearly as large as when the receiver was in the position to absorb the most energy. These lead wires had therefore to be carefully straightened.

7. Another effect mentioned in a former paper[1] as the aging of the vibrator had to be taken into account. The ratio M/C does not remain constant for successive readings when the positions of the receivers remain the same. After the vibrator had been cleaned

[1] Blake and Fountain, l. c.

TABLE I.

Swinging Curve at 15 *cm. August* 7, 1906, 2 *P. M.*
Resistance of main = 1.749 *ohms. Shunts,* 6 *and* 10.
Resistance of check = 1.889 *ohms. Receiver faced.*
Standard value for 0-0-15 *is* 27.80.

Main.			Check.			Ratios.			
Zero.	Turning Point.	Def.	Zero.	Turning Point.	Def.	M/C	Mean.	Standard Mean.	Correct'd Mean.
	0-0-15								
406	44	362	159	72	87	4.17			
404	54	350	157	74	83	4.22	4.20	6.62	27.80
	0-15-15								
408	75	333	158	74	84	3.97—			
407	157	250	158	95	63	3.97	3.97		26.18
	0-345-15								
402	20	382	156	63	93	4.11			
403	40	363	157	69	88	4.13	4.12		27.02
	0-0-15								
402	58	344	155	74	81	4.25			
403	12	391	155	63	92	4.25	4.25	6.54	27.80
	0-60-15								
401	281	120	153	73	80	1.50			
404	304	100	155	88	67	1.49	1.50		9.70
	0-300-15								
401	288	113	152	72	80	1.41+			
402	281	121	152	67	85	1.42+	1.42		9.08
	0-0-15								
402	73	329	149	74	75	4.38			
399	82	317	150	78	72	4.40	4.39	6.33	27.80
	0-30-15								
403	113	290	153	71	82	3.54			
404	98	306	155	69	86	3.56	3.55		22.40
	0-45-15								
402	192	210	154	75	79	2.66			
401	202	199	154	79	75	2.65+	2.66		16.75
	0-0-15								
399	43	356	153	72	81	4.39			
401	36	365	155	73	82	4.45	4.42	6.29	27.80
	0-75-15								
401	360	41	153	84	69	.594			
400	358.5	41.5	153	84	69	.602	.598		3.74
	0-90-15								
400	391	9	154	72	82	.110			
400	390	10	154	66	88	.114	.112		.69
	0-0-15								
398	55	343	152	76	76	4.52			
402	53	349	155	78	77	4.53	4.52	6.15	27.80

and bright surfaces of the balls brought into the sparking position, the ratio M/C would either steadily increase or decrease according to circumstances. But after it had been run for sometime this variation was no longer a steady one. In the former paper we offered a possible explanation of this which seemed to fit all our earlier observations. However, later observations showed that the explanation was far from complete; for when a rigid test was made,

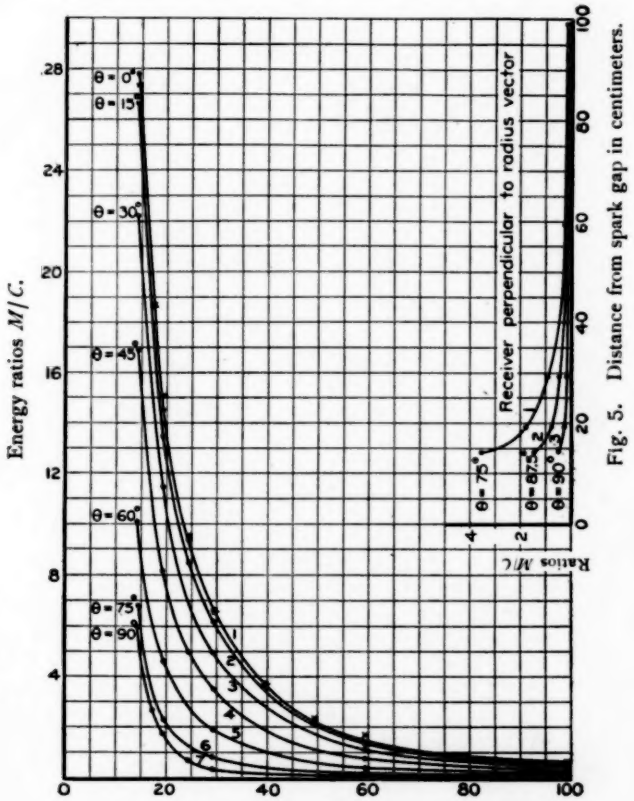

Fig. 5.

Fig. 4. The angle θ opposite the first point on each curve represents the angle between the radius vector and the equatorial plane. The orientation of the receiver is such that maximum energy is received. The crosses indicate points for curve 1 as calculated from the empirical formula in Table II.

the ratio M/C steadily decreased for conditions which seemed identical with those which had previously shown the greatest increase.

At present it seems probable that the wearing away of the balls at the spark gap causes a slight change in the wave-length emitted. Unless the two receivers are tuned to exactly the same wave-length, any change in the wave-length emitted will affect each receiver differently, thus changing the ratio M/C. But whatever the cause may be the effect had to be taken into account. Since

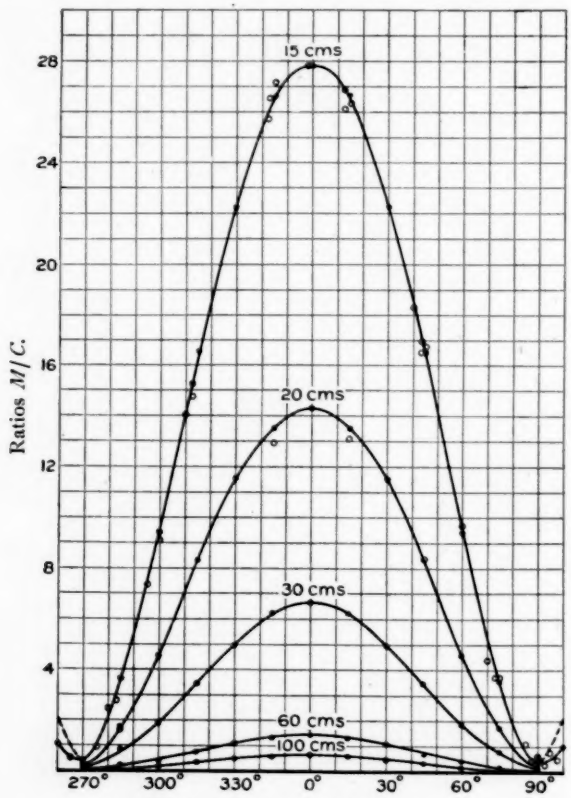

Fig. 6. In each curve the receiver is at a constant distance from the spark-gap. The receiver's axis is always perpendicular to the radius vector. Black dots show mean of all observations for a given angle from the equatorial plane. Circles show individual observations which gave greatest variation from this mean.

the variation showed itself to be a definite linear one for a short time after the vibrator was cleaned, a standard position for the main receiver was chosen and returned to after every fourth setting. The ratios for the intermediate positions were allowed

their percentage increase or decrease as shown by the two values for the standard position. To get a complete curve before the deterioration became variable only two or three readings were taken for each setting.

The effective resistance of the thermal junctions varied with the temperature of the room, therefore the ratios M/C for any one posi-

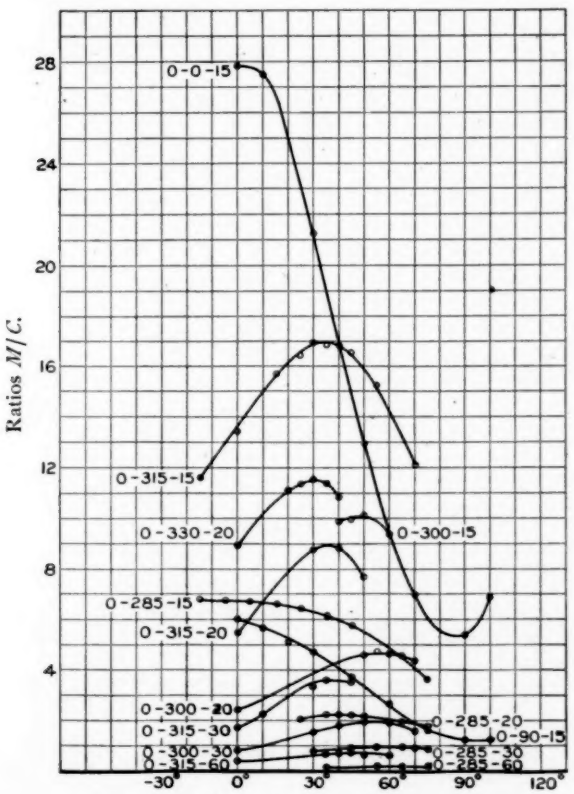

Fig. 7. The orientation of the receiver is changed while its position in the field remains constant.

tion varied from day to day. This ratio also depended upon the galvanometer shunts used. A standard value (27.80) was chosen for the position (0°–0°–15 cm.) of the main receiver. The first figure in denoting the position of the receiver refers to the angle from the horizontal plane measured on the vertical circle H. The second figure denotes the angle from the equatorial plane, that is

the angle θ. The last figure shows the distance (r) in cms. of the middle of the receiver from the spark gap.

The preceeding sample set of observations shows how the various corrections were made to the observed ratios.

EXPERIMENTAL RESULTS.

Curve 1, Fig. 4, is the standard curve upon which all other curves are based. The receiver was kept in the horizontal and equatorial planes while the distance alone was varied. The ratios here determined for the various distances are used as the standard values for all the swinging curves in Fig. 6.

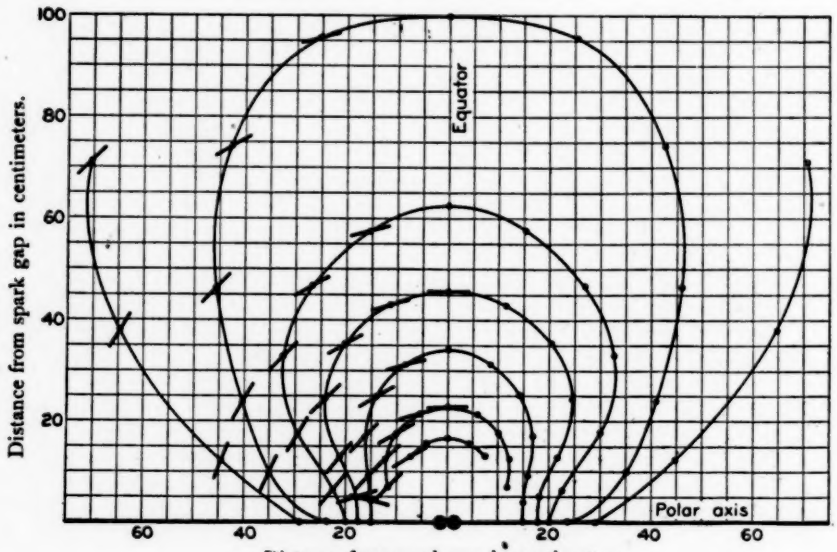

Fig. 8. Equi-energy curves.

The direction curves, Fig. 7, show the direction of the resultant electric force at the various points and the maximum values for such positions. Curves 2 to 6, Fig. 4, were obtained by a combination of the curves in Fig. 7 with those in Fig. 6. The curves in Fig. 5 were obtained directly from those in Fig. 6. The points on the equi-energy curves in Fig. 8 were found by taking arbitrary values for the ratios and ascertaining from the curves in Fig. 4 at what distance those ratios would be found for various angles. That is, Fig. 8 shows the electric field in the horizontal plane

through the vibrator plotted with lines drawn through points which receive the same amount of energy. Each curve runs through the positions at which the energy intensity is twice that for points lying on the curve next beyond it. In one half of the field the orienta-

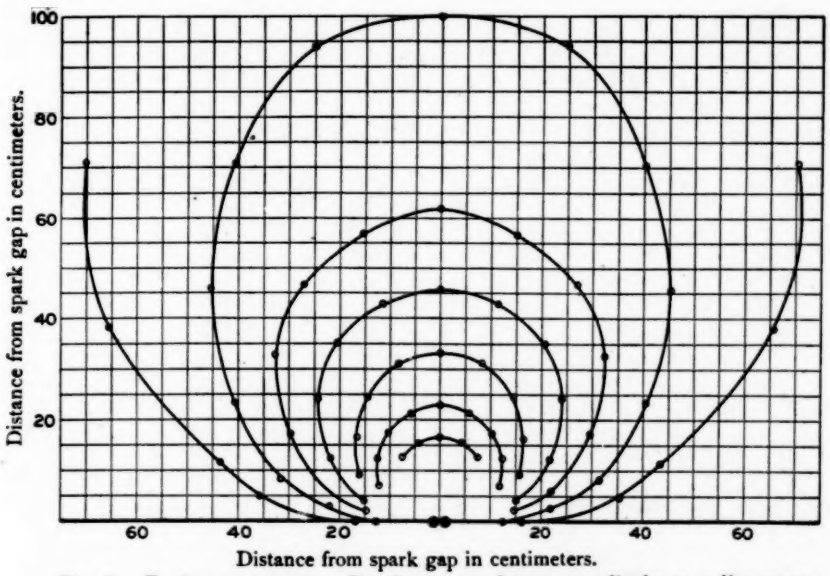

Fig. 9. Equi-energy curves. Receiver everywhere perpendicular to radius vector.

tion of the receiver for maximum energy is shown. In general this direction is at right angles to the radius vector at that point. In the polar direction ($\theta = 90°$), however, this is not true, nor is it

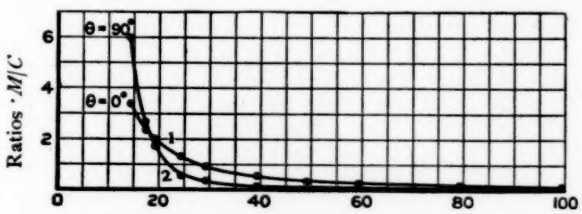

Fig. 10. Radial distance curves.

true for some of the other angles when the receiver is near the vibrator. For example, when $\theta = 75°$ the axis of the receiver must be directed toward the vibrator when $r = 15$ cm., but conforms to the general rule when $r > 50$ cm.

Table II.

Assume $E = K'^2 \frac{\cos^2\theta}{r^2} + k^2 \frac{\sin^2\theta}{r^4}$, *where* $K' = 75.25$, $k = 495$.

Distance.	$\theta = 0°$		$\theta = 15°$		$\theta = 30°$		$\theta = 45°$		$\theta = 60°$		$\theta = 75°$		$\theta = 90°$	
r	Obs.	Cal.	Obs.	Cal.	Obs.	Cal.	Obs.	Cal.	Obs.	Cal.	Obs.	Cal.	Obs.	Cal.
14.4	27.80	27.30	26.65	25.91	22.25	21.84	16.90	16.47	10.11	11.10	6.80	7.11	5.98	5.65
17.4	18.70	18.70		17.36		14.68		10.67		6.65		3.72	2.63	2.64
19.4	14.50	15.04	13.48	14.19	11.45	11.71	8.17	8.38	4.58	5.04	2.22	2.60	1.73	1.71
24.4	9.36	9.51	8.58	8.94		7.31	4.95	5.10		2.89		1.28	.65	.68
29.4	6.63	6.55	6.20	6.15	4.91	5.00	3.44	3.44	1.86	1.88	.85*	.75	.32	.33
39.4	3.74	3.65		3.42		2.77		1.88		.99		.34	.12	.10
49.4	2.03	2.32		2.17		1.75		1.18		.61		.19		.04
59.4	1.43	1.69	1.31	1.50	1.09	1.20	.72	.81	.40	.42	.18	.13	.05	.02
79.4	.92	.90		.84		.68		.45		.23		.07		.008
99.4	.62	.57	.59	.53	.47	.43	.34	.29	.19	.14	.10	.04	.03	.003

In Fig. 9 the equi-energy curves are drawn just as in Fig. 8, except that the receiver has its axis everywhere perpendicular to the radius vector. The curves in Fig. 5 must here be used instead of curves 6 and 7 of Fig. 4.

When $\theta = 0$ and the axis of the receiver is radial the energy absorbed is shown in curve 1, Fig. 10, while curve 2 shows the effect when the receiver is radial in the polar axis ($\theta = 90°$).

The polarization of the waves seems to vary with the distance from the vibrator. Thus at 0–0–15 only 2.7 per cent. as much energy was found when the receiver was perpendicular to the horizontal plane as when it was perpendicular to the equatorial plane. For 0–0–30 the corresponding percentage was 4.6, while at 0–0–60 it was 9.7 per cent.

An empirical formula was found to represent approximately the maximum energy for any point in the field. Table II. shows the corresponding observed and calculated values. The degree of agreement between this formula and the one developed later from purely theoretical considerations will be discussed later.

For points where there is only a very small amount of energy the deflections of the galvanometers due to magnetic and other disturbances render the observations quite inaccurate. That part of Table II. enclosed within the heavy lines includes those observations which are apt to be especially inaccurate. The position marked with a star (*) is not trustworthy because the only readings taken at that point were at a time when the deterioration factor was quite large and irregular. The energy recorded for that part of the field near the vibrator where the radial and the transverse components of the forces have about the same effect will probably be too small, for the receiver cannot be turned so as to get the full effect of either force.

Theoretical Discussion.

Assuming that the field about a Righi vibrator is similar to the field about a theoretical electric doublet the forces in the field may be represented as has been done by A. E. H. Love.[1] That is the transverse component of the electric force is

[1] A. E. H. Love, Proc. Royal Soc., Vol. 74, p. 73, 1904.

$$F = \frac{\cos\theta}{r^3} A e^{-\frac{\nu}{\lambda}(ct-r)} \left[\left(1 - \frac{r\nu}{\lambda} + \frac{r^2(\nu^2 - 4\pi^2)}{\lambda^2} \right) \sin \frac{2\pi}{\lambda}(ct - r + \varepsilon) + \frac{2\pi r}{\lambda} \left(1 - \frac{2r\nu}{\lambda} \right) \cos \frac{2\pi}{\lambda}(ct - r + \varepsilon) \right], \quad (1)$$

when $ct > r$, and when $ct < r$ it is

$$F = \frac{\cos\theta}{r^3} A \sin \frac{2\pi\varepsilon}{\lambda}. \quad (2)$$

Where θ is the angle between the radius vector and the equatorial plane, r the distance along the radius vector, A a constant depending upon the amplitude of vibration, ν the damping constant, λ the wave-length, c the velocity of radiation, t the time from the beginning of the oscillation, and ε a constant depending upon the phase between the electric and magnetic forces.

The radial component of the electric force is

$$R = \frac{2\sin\theta}{r^3} A e^{-\frac{\nu}{\lambda}(ct-r)} \left[\left(1 - \frac{r\nu}{\lambda} \right) \sin \frac{2\pi}{\lambda}(ct - r + \varepsilon) + \frac{2\pi r}{\lambda} \cos \frac{2\pi}{\lambda}(ct - r + \varepsilon) \right], \quad (3)$$

when $ct > r$, and

$$R = \frac{2\sin\theta}{r^3} A \sin \frac{2\pi\varepsilon}{\lambda}, \quad (4)$$

when $ct < r$.

The transverse component of the magnetic force is

$$H = -\frac{\cos\theta}{r^2} A e^{-\frac{\nu}{\lambda}(ct-r)} \left[\left(\frac{\nu}{\lambda} - \frac{r(\nu^2 - 4\pi^2)}{\lambda^2} \right) \sin \frac{2\pi}{\lambda}(ct - r + \varepsilon) - \frac{2\pi}{\lambda} \left(1 - \frac{2r\nu}{\lambda} \right) \cos \frac{2\pi}{\lambda}(ct - r + \varepsilon) \right], \quad (5)$$

when $ct > r$, but vanishes for $ct < r$.

The radial component of the magnetic force is everywhere equal to zero.

If these represent the true values for the forces, the current produced by them at any point in a conductor placed in the field can be determined. Consider the receiver in the equatorial plane with its axis parallel to the axis of the vibrator. The forces in the field

are practically the same about every point in the receiver. So the average force about the receiver is proportional to the force at any particular point in it, if it be more than 14 cm. from the vibrator. The current at any point in the receiver will be proportional to the forces about any particular point in it. That is, the rate of change with the distance is practically the same for all the forces about the receiver. Therefore the current at the thermal junction will be proportional to the forces in the immediate neighborhood of the junction. The deflections of the galvanometers will be proportional to the heat developed at the thermal junction; that is, proportional to the square of the current at the junction.

$$\therefore \text{Heat} \propto \int_0^\infty i^2 dt \propto \int_{r/c}^\infty (F^2 + H^2) dt.$$

Since $\tan 2\pi\varepsilon/\lambda = 2\pi/\nu$,[1]

$$\int_{r/c}^\infty (F^2+H^2)dt = \frac{\cos^2\theta}{r^2} A^2 \frac{2\pi^2}{c\nu\lambda^3}(4\pi^2+\nu^2)\left[1 - \frac{2\nu\lambda^3}{r^3(4\pi^2+\nu^2)^2} + \frac{\lambda^4(4\pi^2+5\nu^2)}{2r^4(4\pi^2+\nu^2)^3}\right]. \quad (6)$$

Within the region explored the last two terms in the brackets could never be larger than .0002. Therefore the electromagnetic energy will be approximately proportional to $\cos^2\theta/r^2$. This is identical with the empirical equation in Table II.

Let us now consider what effect the radial forces will have upon the receivers. If the main receiver is in the polar region ($\theta = 90°$) with its axis parallel to the radius vector the only force acting along its axis will be the radial component of the electric force. Again assuming that the current at the junction is proportional to the forces immediately surrounding that point, the energy absorbed will be proportional to

$$\int_{r/c}^\infty R^2 dt = \frac{A^2\pi}{2c\nu\lambda} \frac{\sin^2\theta}{r^4}\left[1 - \frac{\lambda}{r}\frac{4\nu}{(4\pi^2+\nu^2)} + \frac{\lambda^2(4\pi^2+5\nu^2)}{r^2(4\pi^2+\nu^2)^2}\right]. \quad (7)$$

Calculating the value of ν from the interference curves obtained by Willard and Woodman[1] for waves 20 cm. long it seems probable that the value of ν is not greater than .15. Substituting this value

[1] See Love, l. c.

in (7) the last two terms in the brackets become larger than .02 for $r = 15$ cm. but only .001 when $r = 30$ cm.

The values in the last column of Table II. were calculated from the equation $E = k^2/r^4$, where k was 495. If the last two terms of (7) be introduced and k be given the value 489 the figures for the last column of Table II. become respectively, 5.72, 2.65, 1.71, .68, .32, .10, .04, .02, .008, .003. These show a slightly better agreement with the observed values. The value assumed for ν is prob-

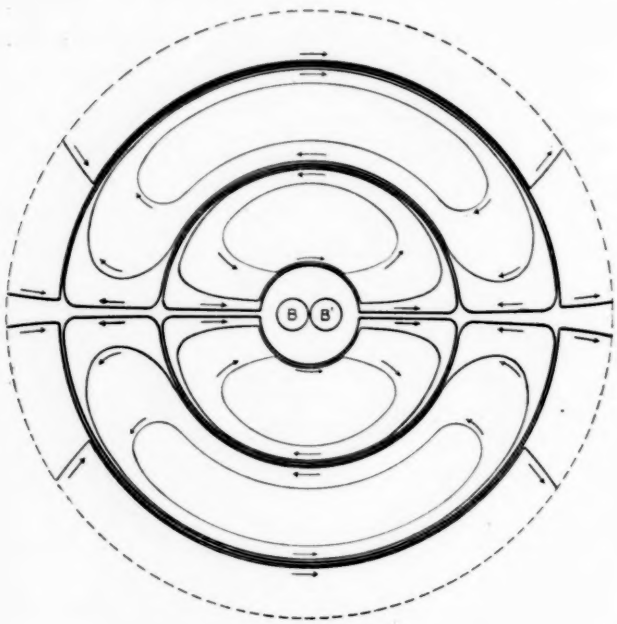

Fig. 11. Lines of electric force.

ably too large. A smaller value would give still better agreement.

The assumption that the current at the junction is proportional to the forces at that point is probably not strictly true. When $r = 15$ cm. the value of R at the center of the receiver is certainly not proportional to the average value of R for the entire length of the receiver. As the charges begin to build up at the ends of the receiver the forces of their mutual repulsion increase so as to oppose the force R with increasing effectiveness for increasing distance from the center of the receiver. To compute the

effect of these repulsive forces would require a better knowledge of the number of free electrons in the copper strips and their behavior under oscillating forces than the authors have at present. Some idea of the extent to which these forces counteract the effect of the forces in the field near the ends of the receiver may be seen by neglecting these forces and computing the ratio of energy absorbed when $r = 15$ cm. to that absorbed when $r = 30$ cm. There is a 40 per cent. variation from the observed ratio. The error involved for the corresponding distances in assuming that the current at the junction is proportional to the value of R at that point is probably less than 4 per cent.

The physical significance of the radial force may be seen from Fig. 11, where the lines of force are plotted for

$$t = \frac{\frac{3}{2}\lambda - \varepsilon}{c}.$$

The three values for the intensity of the force here plotted are to each other as 10,000 : 100 : 10. The dotted circle is $\frac{3}{2}\lambda$ from the spark-gap. Lines are drawn up to a distance of $\frac{1}{4}\lambda$ although the formula used probably does not hold so near to such a vibrator. As the energy flows into the polar region from the two halves of the field the electric forces in the two approaching waves will always be in the same relative direction. But since the direction of the energy flow is opposite in the two cases the magnetic forces must be opposed to each other. Therefore if the two halves of the field were symmetrical there would be at any point on the polar axis a varying electric force, but the resultant magnetic force would always be equal to zero. Prof. G. F. FitzGerald [1] made this case the basis of an article on the "Longitudinal Component in Light."

When the receiver is in the polar region ($\theta = 90°$) with its axis perpendicular to the polar axis the energy absorbed varies approximately as $1/r^4$ and therefore might suggest that it was due to some effect of the force R upon the lead wires. Varying the positions of the lead wires produced only slight changes in the energy absorbed. The transverse components of the electric and magnetic forces become equal to zero when $\theta = 90°$. But the ends of a receiver 8

[1] Phil. Mag., Vol. 42, p. 266, 1896.

cm. long reach out on either side into the field where there is a force component toward the middle of the receiver. This is readily seen from Fig. 11. The forces at the two ends would at the same instant be both directed toward the center so that the only flow must be from the ends of the receiver past the thermal junction into the capacity of the galvanometer circuit and out again. If such a current were proportional to the forces at the ends of the receiver the heat developed at the junction would vary approximately as shown in curve 3, Fig. 5. The whole effect is so small that its exact nature is very difficult to determine.

When θ is 0° and the receiver radial the energy absorbed varies as $1/r^2$, that is in the same way as when the receiver is perpendicular to the radius vector. Varying the lead wires failed to reveal the cause for an absorption of energy as great as shown by curve 1, Fig. 10. The radial component of the electric force should be zero for this position, and the axis of the receiver is perpendicular to both the electric and magnetic forces. No satisfactory explanation for this phenomenon has yet been found.

CONCLUSIONS.

The results of the present investigation seem to indicate:

1. That in computing the total energy in the field about a Righi vibrator it is necessary to consider the radial component of the electric force as well as the transverse components of the electric and magnetic forces.

2. That the electromagnetic waves spread out from the spark gap in such a manner that the energy at any point is approximately proportional to the square of the cosine of the angle between the radius vector and the equatorial plane and inversely proportional to the square of the distance from the spark gap.

3. That the energy due to the radial component of the electric force varies approximately as the square of the sine of the angle between the radius vector and the equatorial plane and inversely as the fourth power of the distance from the spark gap.

4. That in order to receive the maximum energy the thermal element of the Klemenčič type should everywhere be perpendicular to the radius vector except near the poles, where it should be parallel to the radius vector.

5. That the thermal element used in these experiments when placed in the equatorial plane with its axis parallel to the direction of propagation of the electromagnetic waves absorbs more than 14 per cent. as much energy as when it is in the resonating position.

6. That this type of receiver absorbs a small amount of energy when it is on the polar axis and perpendicular to it, and therefore perpendicular to the radius vector.

7. That two such receivers in the same field greatly influence each other.

8. That a Righi vibrator does not continue to give out wave-trains of the same strength or character as shown by its "deterioration" with use.

9. That the waves from a Righi vibrator appear to become less plane polarized as they proceed from the vibrator.

10. That a theory based upon the assumption that the waves from a Righi vibrator are identical with those from a vibrating "electric doublet" and also that the thermal element of the Klemenčič type acts as a "point receiver" holds with a fair degree of accuracy up to within $\frac{1}{4}\lambda$ from the vibrator.

We wish to thank Mr. J. B. Blanchard and others for their assistance in taking measurements. To Professor Hallock and Professor Wills we are indebted for their suggestions and friendly interest. Our acknowledgments are especially due to Professor E. F. Nichols who suggested the problem and some of the methods for solving it.

PHŒNIX PHYSICAL LABORATORIES,
COLUMBIA UNIVERSITY,
April, 1907.

A STUDY OF THE REVERSIBLE PENDULUM.[1]

PART I: THEORETICAL CONSIDERATIONS.

BY JOHN C. SHEDD AND JAMES A. BIRCHBY.

IF the entire mass of a compound pendulum be M and its radius of gyration about the centroid of the pendulum be a, its moment of inertia about the centroid is Ma^2 and its moment of inertia about a parallel axis at a distance z is $M(a^2 + z^2)$. If this second axis be taken as the center of support of the pendulum, then its equivalent length is

$$l = z + \frac{a^2}{z}. \tag{1}$$

Solving for z the expression is had,

$$z = \tfrac{1}{2}(l \pm \sqrt{l^2 - 4a^2}). \tag{2}$$

From this it appears that for any given equivalent length there are in general two possible positions of the supporting knife edge with reference to the centroid, *i. e.*,

$$\left.\begin{aligned} z_1 &= \tfrac{1}{2}(l + \sqrt{l^2 - 4a^2}) \\ z_2 &= \tfrac{1}{2}(l - \sqrt{l^2 - 4a^2}) \end{aligned}\right\}. \tag{3}$$

If the pendulum be provided with two knife-edges k_1 and k_2 then the equivalent lengths from the respective knife-edges are

and

$$\left.\begin{aligned} l_1 &= z_1 + \frac{a^2}{z_1} \\ l_2 &= z_2 + \frac{a^2}{z_2} \end{aligned}\right\}. \tag{4}$$

[1] Read before joint meeting of the Am. Phys. Soc. and Section B of the A. A. A. S., New York, December 29, 1906.

These lengths will in general be different in value but it is of course possible that $l_1 = l_2$.

If this be the case, then $a^2 = z_1 z_2 (z_1 - z_2)/(z_1 - z_2) = z_1 z_2$, provided (in general) that $z_1 \neq z_2$. It is thus seen that the radius of gyration is the mean proportional between the segments (z_1 and z_2) into which the centroid divides the distance between the knife-edges.

Substituting the value $z_1 z_2$ for a^2 in the expression for l_1 and l_2 it is seen that

$$l_1 = l_2 = z_1 + z_2,$$

showing that the equivalent length now becomes the distance between knife-edges. Since a is the mean proportional between z_1 and z_2 it is apparent that its value cannot exceed

$$\tfrac{1}{2}(z_1 + z_2), \quad \text{or} \quad \tfrac{1}{2}L.$$

If this condition with respect to a be not fulfilled, the same period cannot be secured from both knife-edges, and the pendulum cannot be brought into proper adjustment.

The fact that it is possible to realize the condition $l = z_1 + z_2$ was made use of by Captain Kater in 1817, and constitutes the basis of the reversible pendulum bearing his name. Since then the general theory of the compound pendulum up to the point that Kater applied it has been completely worked out and may be found in various works on mechanics. This also includes a discussion of precautions and corrections necessary when the pendulum is used as an instrument of precision. In a recent article[1] Dr. R. R. Tatnall has somewhat extended the theoretical discussion to cover a number of important points. In the present paper it is proposed to discuss the equation of the pendulum from both the analytical and graphical standpoints, and also the various possible methods by which the pendulum may be used. Under the latter head several novel modifications will be pointed out.

From equation (1), together with the definition of z, it is seen that l (and therefore the period of the pendulum), is independent of the direction in which z is measured, provided only that it lie in a plane passing through the centroid of M and normal to the axis

[1] PHYSICAL REVIEW, VOL. XVII., 1903. See also A. O. Allen, Math. Gazette' Vol. III., 1906.

about which the moment of inertial is taken. Two circles may thus be drawn about the centroid (Fig. 1, C) having radii equal to z_1 and z_2 respectively, and all points on either of these circles taken as points of suspension will give the same equivalent length, and hence the same period for the pendulum. The values of the radii of these circles are given by equations (3). In practice the points A, B, D and E are the only points realizable and it is seen that for every compound pendulum there are four such points having the same equivalent length and the same period. The value of this common period is,

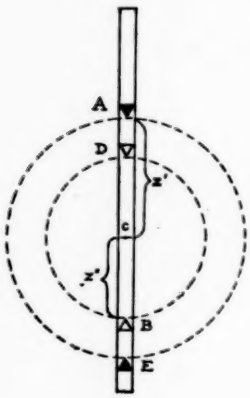

Fig. 1.

$$t = \pi \sqrt{\frac{l}{g}} \qquad (5)$$

In general experimental work the point A is taken as fixed by one knife-edge and the other knife-edge is moved until experimentally located at B. Under this condition $l = z_1 + z_2$ which equals the distance between knife-edges. Or both knife-edges are fixed and the movable masses of the pendulum are adjusted until the above condition is realized. This is signalized by an equality in the period taken from either knife-edge. Let us suppose, however, that in the adjustment the knife-edges fall at B and D respectively. It is evident that the distance between knife-edges is no longer the equivalent length of the pendulum, being now too short; similarly, if the knife-edges fall at A and E the distance between knife-edges is too great. Under these conditions the relation between the distance (L) between knife-edges and the equivalent length (l) is given by the equation

$$L = l \pm \sqrt{l^2 - 4a^2}. \qquad (6)$$

A special case arises when $l = 2a$ and $L = l$; under this condition it is seen from (3) that $z_1 = z_2$ and that the centroid is located midway between the knife-edges. The further condition is involved that $a = \frac{1}{2}L$. Graphically this condition is expressed in Fig. 1 by the coalescence of the two circles. It is thus seen that it is pos-

sible (contrary to the general statements found in text-books) to have the centroid midway between the knife-edges and yet to bring the pendulum into adjustment.

The Equation of the Pendulum. — Let Fig. 2 represent a reversible pendulum having fixed knife-edges k_1 and k_2, and a movable mass m, all other parts being fixed.

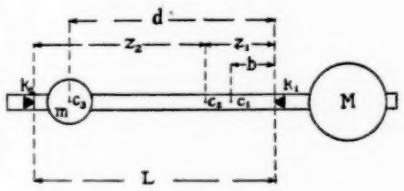

Fig. 2.

Let M = the mass of the pendulum aside from m,

b = the distance from k_1 to C_1, the centroid of M,

z_1 = the distance from k_1 to C_2, the centrod of $M + m$,

z_2 = the distance from k_2 to C_2, the centroid of $M + m$,

d = the distance from k_1 to C_3 the centroid of m,

$I_b = Ma^2$ = the moment of inertia of M about C_1,

$I = (M + m)a^2$ = the moment of inertia of $M + m$ about C_2,

$I_m = m\beta^2$ = the moment of inertia of m about C_3,

a = the radius of gyration of $M + m$ about C_2.

$R = m/M$,

Then $a^2 = I/(M + m)$; whence from equation (1), $l_1 = z_1 + I/z_1(M + m)$. But since $I = I_b + I_m + M(z_1 - b)^2 + m(d - z_1)^2$,

therefore

$$l_1 = z_1 + \frac{I_b + I_m + M(z_1 - b)^2 + m(d - z_1)^2}{z_1(M + m)}; \qquad (7)$$

Whence

$$t = \pi\sqrt{\frac{\bar{l}}{g}} = \pi\left\{\frac{z_1}{g} + \frac{I_b + I_m + M(z_1 - b)^2 + m(d - z_1)^2}{gz_1(M + m)}\right\}^{\frac{1}{2}}. \qquad (8)$$

From the principle of moments, $M(z_1 - b) = m(d - z_1)$, or

$$z_1 = (Mb + md)/(M + m) = \frac{b + Rd}{1 + R}. \qquad (9)$$

Squaring both sides of equation (8) and substituting from equation (9) it is seen that,

$$\frac{gt_1^2}{\pi^2}(b + Rd) = Rd^2 + b^2 + a^2 + R\beta^2. \qquad (10)$$

In equations (8) and (10) the pendulum is supposed to be suspended from k_1. If now the pendulum be suspended from k_2 the terms b, z_1, and d appear as $L-b$, $L-z_1$, and $L-d$, so that equation (10) becomes,

$$\frac{gt_2^2}{\pi^2}[L(1+R)-(b+Rd)] = R(L-d)^2+(L-b)^2+\alpha^2+R_i\beta^2. \quad (11)$$

Equations (10) and (11) may be called the equations of the reversible pendulum.

A discussion of these equations constitutes the problem of the reversible pendulum. In this discussion there is a choice of terms that may be taken as variables. The pairs of variables that will be investigated are (1) t and d, (2) t and β, (3) t and R. Inspection shows that b is not independent of d, and that a change in α has essentially the same effect as a change in β, so that the above pairs are sufficient to cover the problem.

DIVISION I. t AND d TAKEN AS VARIABLES.

If in equations (10) and (11) the origin be moved to

$$\left(-\frac{b}{R},\ 0\right) \text{ and } \left(\frac{L}{R}-\frac{b}{R}+L,\ 0\right)$$

respectively, and in (11) d be changed to $-d$, the equations reduce respectively to

$$\left.\begin{aligned} t_1^2 d &= Ad^2 + Bd + C \\ t_2^2 d &= A'd^2 + B'd + C' \end{aligned}\right\} \quad (12)$$

where the constants have the following values:

$$A=\frac{\pi^2}{g};\ B=-\frac{2\pi^2 b}{Rg};\qquad C=\frac{\pi^2}{g}\left[\frac{(1+R)b^2}{R^2}+\frac{\alpha^2}{R}+\beta^2\right],$$

$$A'=\frac{\pi^2}{g};\ B'=\frac{2\pi^2(L-b)}{Rg};\ C'=\frac{\pi^2}{g}\left[\frac{(1+R)(L-b)_2}{R^2}+\frac{\alpha^2}{R}+\beta\right].$$

Applying the criterion suggested by Sir Isaac Newton for cubic curves of the form of equations (12) it is seen that the roots of both $Ad^2+Bd+C=0$, and $A'd^2+B'd+C'=0$ are imaginary, since

$$B^2 - 4AC = -\frac{4\pi^4}{g^2}\left[\frac{b^2 + a^2}{R} + \beta^2\right]$$

and

$$B'^2 - 4A'C' = -\frac{4\pi^4}{g^2}\left[\frac{(L-b)^2 + a^2}{R} + \beta^2\right],$$

both of which quantities are necessarily negative. Therefore both (10) and (11) are of the fifty-third species in Newton's classification. The form of this curve is represented in Fig. 3. Since the right-

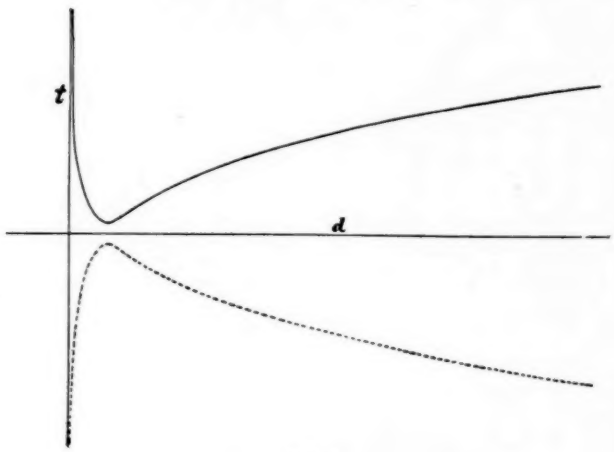

Fig. 3.

hand side of (10) is always positive, t_1 is real for any value of d which makes the left side positive. The condition for real values of t_1 then becomes:

$$d > -\frac{b}{R}. \qquad (13)$$

It follows that if b is positive t_1 is real for all positive values of d and for negative values numerically smaller than b/R. If b is negative t_1 is real only when d is positive and numerically greater than b/R. For $d = -b/R$ t_1 is infinite, and the curve has an asymptote parallel to the axis of t at this point. (In Fig. 3 the origin has been shifted so that the Y axis is this asymptote.) In the pendulum a negative value of d means that the centroid of the movable mass is above k_1, and when it is at a distance b/R above k_1 the centroid of the whole pendulum is at the supporting knife-edge. If

the movable mass be still further above k_1 the pendulum is unstable, a condition indicated by an imaginary value for t_1 in equation (10).

In like manner t_2 in (11) is real for,

$$d < L + (L-b)/R \tag{14}$$

It is easy to see that for the limiting value of d here given the centroid of the whole pendulum is at k_2. The line $d = L + (L - b)/R$ is an asymptote to the curve, the value of t_2 approaching infinity as d approaches this value.

Differentiating equation (10) and regarding t_1 as the dependent and d_1 the independent variable the expression is obtained :

$$\frac{dt_1}{d(d)} = \frac{R(2\pi^2 d - gt_1^2)}{2gt_1(Rd + b)}. \tag{15}$$

This is zero when

$$\frac{gt_1^2}{\pi^2} = 2d.$$

Eliminating t_1 between (10) and this equation the values of d for maxima or minima are had. These values are,

$$d = -\frac{b}{R} \pm \sqrt{\frac{b^2}{R^2} + \frac{b^2 + a^2}{R} + \beta^2} \tag{16}$$

The negative value of the radical gives an imaginary value for t_1 since it puts d beyond the limit assigned by (13). Hence there is but one such real point, and it may be shown that this corresponds to a minimum point for positive values, and a maximum for negative values of t_1. Since we are concerned only with conditions that may be experimentally realized, positive values of t_1 are alone considered.

In a similar manner, by differentiating equation (11) the expression is had :

$$\frac{dt_2}{d(d)} = \frac{R[gt_2^2 - 2\pi^2(L - d)]}{2gt_2[R(L - d) - (L - b)]}. \tag{17}$$

From which we get :

$$d = \frac{L(R + 1) - b}{R} \pm \sqrt{\frac{(L - b)^2}{R^2} + \frac{(L - b)^2 + a^2}{R} + \beta^2}, \tag{18}$$

for points of maxima or minima.

The positive value of the radical gives an imaginary value to t_2 since it throws d outside of the limit assigned by (14). The negative value of the radical gives a minimum for positive values of t_2, and as before it is seen that there is but one such real point.

It is interesting to note that for these minimum points on the two curves the values of t_1 and t_2 are given by the expressions

$$t_1 = \pi\sqrt{\frac{2d}{g}} \quad \text{and} \quad t_2 = \pi\sqrt{\frac{2(L-d)}{g}}, \tag{19}$$

thus showing that at these points the equivalent lengths are respectively equal to $2d$ and $2(L-d)$. If in addition to this we specify that $t_1 = t_2$ then it is seen that $L - d = d$, or $d = \frac{1}{2}L$. This involves the condition that the two curves shall be tangent to each other on the horizontal line

$$t = \pi\sqrt{\frac{L}{g}}.$$

An interesting case arises when $b = \frac{1}{2}L$. Equations (16) and (18) become

$$\left.\begin{aligned} d' &= -\frac{L}{2R} \pm \sqrt{\frac{L^2}{4R^2} + \frac{L^2 + 4a^2}{4R} + \beta^2}, \\ d'' &= L + \frac{L}{2R} \pm \sqrt{\frac{L^2}{4R^2} + \frac{L^2 + 4a^2}{4R} + \beta^2}. \end{aligned}\right\} \tag{20}$$

The negative value of the radical in the upper equation is precluded by equation (13), as is also the positive value in the lower equation by (14). Applying this fact, it will now be seen that $d' + d'' = L$, or in words: *when the centroid of the fixed part of the pendulum is placed midway between the knife-edges then the sum of the abscissas of the minimum points of the two curves equals the distance between the knife-edges.*

Of the nine possible intersections of two cubic curves, in the present case three are imaginary or at infinity, three belong to the condition that t is negative, and three belong to positive values of t, and can hence be experimentally realized.

Subtracting equation (10) from (11) the locus ot the line passing through the intersection points is obtained. This gives:

$$\left(L - \frac{gt^2}{\pi^2}\right)[L - 2b + R(L - 2d)] = 0. \qquad (21)$$

From this two straight lines are obtained. The first is the line

$$t = \pi\sqrt{\frac{L}{g}}, \qquad (22)$$

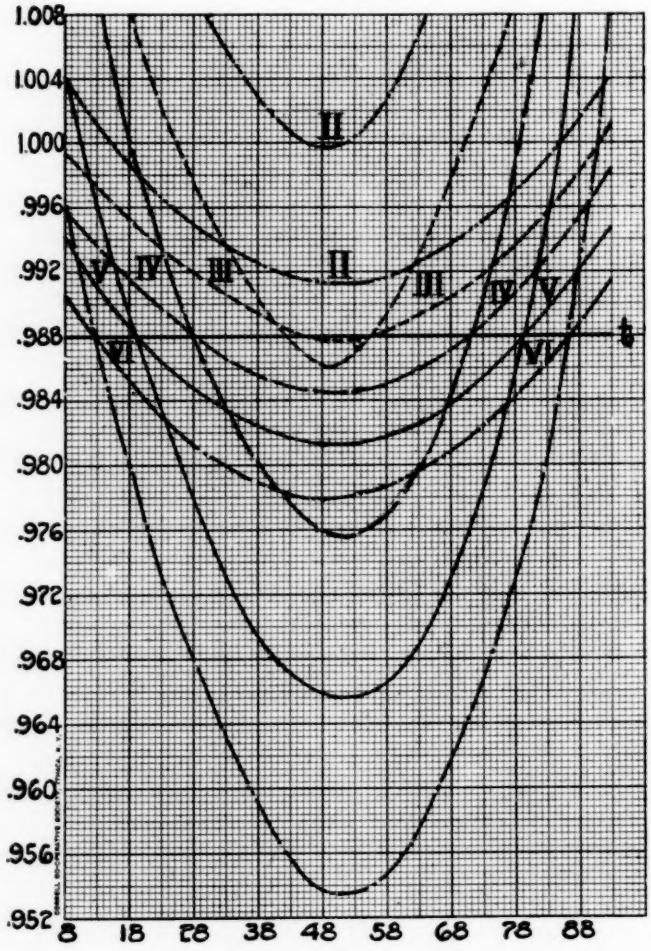

Fig. 4.

and inspection shows that two of the intersections lie on this line which is parallel to the X-axis and at a distance $\pi\sqrt{L/g}$ above it. It is one or other of these intersections (see Fig. 4) that is generally

sought for in experimental work and is the condition used by Kater, *i. e.*, that the equivalent length of the pendulum shall equal L. Eliminating t from equations (10) and (22) the values of the abscissas of the intersection points are had. This gives

$$d = \tfrac{1}{2}L \pm \sqrt{L^2 - 4\left[\frac{a^2}{R} + \beta^2 - \frac{b}{R}(L-b)\right]}. \qquad (23)$$

This expression shows that the intersection points lie symmetrically with respect to the point midway between k_1 and k_2. As seen above, the curves became tangent, *i. e.*, the intersection points coincide at the point $(\frac{1}{2}L, \pi\sqrt{L/g})$. Under this condition the radical in (23) is equal to zero and

$$R(\beta^2 - \tfrac{1}{4}L^2) = b(L-b) - a^2. \qquad (24)$$

The second line obtained from (21) is:

$$d = \frac{L(R+1) - 2b}{2R}. \qquad (25)$$

This line is parallel to the axis of Y and passes through the third intersection of the two curves. This third intersection marks the condition when the centroid of the whole pendulum falls midway between k_1 and k_2. It may fall on either side of the first two intersections or between them. Again, it may coincide with either of them or all three may coincide. Substituting the value of d in (25) into equation (10) the value of t is found to be:

$$t = \left[\frac{\pi^2}{2g}\,\frac{(L-2b+RL)^2 + 4R(b^2 + a^2 + R\beta^2)}{LR(R+1)}\right]^{\frac{1}{2}}. \qquad (26)$$

This is the common period from either knife-edge on the condition that $z_1 = z_2$. Whether this intersection fall below, upon, or above the line

$$t = \pi\sqrt{\frac{L}{g}}$$

depends upon whether

$$\frac{(L-2b)^2}{R} + \frac{4(a^2 + R\beta^2)}{1+R}$$

is less than, equal to, or greater than L^2. The coördinates of the three intersection points are as follows:

$$
\left.
\begin{aligned}
&(1)\quad \begin{cases} d_1 = \tfrac{1}{2}L + \sqrt{\dfrac{RL^2 + 4(L-b)b - 4(a^2 + R_i\beta^2)}{4R}}, \\ t_1 = \pi\sqrt{\dfrac{L}{g}}, \end{cases} \\
&(2)\quad \begin{cases} d_2 = \tfrac{1}{2}L - \sqrt{\dfrac{RL^2 + 4(L-b)b - 4(a^2 + R_i\beta^2)}{4R}}, \\ t_2 = \pi\sqrt{\dfrac{L}{g}}, \end{cases} \\
&(3)\quad \begin{cases} d_3 = \dfrac{L(R+1) - 2b}{2R}, \\ t_3 = \left[\dfrac{\pi^2}{2g}\dfrac{(L - 2b + RL)^2 + 4R(b^2 + a^2 + R_i\beta^2)}{LR(R+1)}\right]^{\frac{1}{2}}. \end{cases}
\end{aligned}
\right\} \quad (27)
$$

The conditions required for a coincidence of any two of the points are readily had by assuming the coördinates of 1 and 2, 1 and 3, and 2 and 3 successively equal. This gives three cases with a fourth where all three points coincide.

Case 1. $d_1 = d_2 = \frac{1}{2}L$. — The term under the radical here equals zero. Setting this equal to zero and solving for b the expression is had

$$b = \tfrac{1}{2}L \pm \sqrt{\tfrac{1}{4}L^2(R+1) - a^2 - R_i\beta^2}. \qquad (28)$$

These values of b are seen to be symmetrical with respect to the value $d = \frac{1}{2}L$. If $b = \frac{1}{2}L$ then the centroid of the whole pendulum is at this point (since $d = \frac{1}{2}L$) and the three intersections coincide. Keeping the first two interesections together, the third will move to the right or left according as the + or − sign holds good in equation (28).

Cases 2 and 3. — The condition for these cases may be expressed by the equation

$$L = 2b \pm 2R\sqrt{K}$$

where

$$K = \frac{RL^2 + 4(L-b)b - 4(a^2 + R_i\beta^2)}{4R}.$$

Solving for the value of b we find that

$$b = \tfrac{1}{2}L \pm \sqrt{\frac{RL(R+1) - 4R(a^2 + R\beta^2)}{R+1}}. \qquad (29)$$

The negative sign is to be used for case 2 and the positive sign for case 3. It is here seen that, beginning with the value $b = \frac{1}{2}L$ (when all three intersections coincide) the tangency point recedes to the right (case 3) or left (case 2) as b increases or decreases. It will be noticed that a given change in the variable in equation (29) will produce an equal change in the value of b under both case 2 and case 3. In a sense, therefore, the points of tangency may be said to be symmetrically located with respect to the value $b = \frac{1}{2}L$.

Case 4. — The condition reduces to the condition that $d = \frac{1}{2}L$ and also $b = \frac{1}{2}L$. This gives a value for R, $R = (4a^2 - L^2)/(L^2 - 4\beta^2)$. Since R must be positive, this gives $a \gtrless L/2 \gtrless \beta$. If $R = 0$, $L^2 = 4a^2$, a result given by equation (6) if $L = l$. Finally it may be noted that if $b = \frac{1}{2}L$ (but $L - 2b \mp 2R\sqrt{K} \neq 0$) then the third intersection falls on the line $d = \frac{1}{2}L$ but below the line $t = \pi\sqrt{L/g}$.

In the literature of the subject it is generally assumed that the centroid of the pendulum must not be allowed to fall midway between k_1 and k_2. From the preceding it is seen that this is permissible and that the value of t is then given by equation (26); and, further, that this expression may reduce to the usual one given in equation (22). A glance at equation (26) is sufficient, however, to convince one that this equation cannot be used in practice; also, when the three points of intersection coincide and equation (26) reduces to (22) the curves become osculating curves and the intersection point is not well defined, hence the condition $z_1 = z_2$ is in practice on all accounts to be avoided.

Division II. t and β Taken as Variables.

In order that a change in β shall not produce a change in d also, the pendulum must be modified as shown in Fig. 5, which represents a pendulum with two knife-edges k_1 and k_2 and with two movable masses m_1, m_2 which can be so moved along the bar of the pendulum that their centroid (c_3) shall remain fixed. The distance of these masses from c_3 may be called q and it is seen that the moment of inertia of the movable mass becomes $2m\beta_0^2 + 2mq^2$ in

which $m\beta_0^2$ is the moment of inertia of m_1, m_2, each about its own centroid. Equation (10) may now be written

$$gt^2/\pi^2(b + Rd) = Rq^2 + R\beta_0^2 + Rd^2 + b^2 + a^2. \qquad (30)$$

t and q are the variables in this equation and it is therefore of the form

$$At^2 = Bq^2 + C \qquad (31)$$

where

$$A = q/\pi^2(b + Rd),\ B = R, \quad \text{and} \quad C = R(\beta_0^2 + d^2) + b^2 + a^2.$$

The constants A, B, and C are necessarily positive.

Equation (31) is the equation of a central hyperbola whose axis is the axis of Y. Since negative values of t and q are not permissible, that part of the curve falling in the first quadrant is alone to be considered. From equation (31) it appears that t is a minimum for q equals zero, varies in the same sense as C and in the opposite sense as A. The equation of the asymptotes is,

$$t = \pm \frac{B}{A} q = \pm \frac{R\pi^2 q}{g(b + Rd)}, \qquad (32)$$

which shows that the slope of the asymptotes varies inversely as

$$\left(\frac{b}{R} + d\right).$$

If the pendulum be swung from the second knife-edge equation (27) becomes :

$$\frac{gt^2}{\pi^2}[L - b + R(L - d)] = Rq^2 + R\beta_0^2 + R(L - d)^2 + (L - b)^2 + a^2 \qquad (33)$$

which is of the same form as (28) but with changed values for the constants.

If in equations (27) and (30) y be substituted for t^2 and x for q^2 as the variables, the equations take the form,

$$Ay = Bx + C \quad \text{and} \quad A'y = B'x + C' \qquad (34)$$

in which A, B, and C and A', B', and C' have the same values as before. Equations (34) are the equations of two straight lines

having different slopes. The intersections of all the pairs of lines represented by (34) lie on the line,

$$y = \frac{C - C'}{A - A'} = \frac{\pi^2 L}{g}. \tag{35}$$

The condition for a realizable intersection is that the coördinates (x_1, y_1) of the point of intersection both be positive. y_1 necessarily is positive, from (35); and substituting this value of y_1 in the equations of (34) shows that x_1 is positive, provided

$$L > (I_b + I_{2m} + Mb^2 + 2md^2)/(Mb + 2md).$$

This condition may be expressed thus: *x is positive when the distance between the knife-edges is greater than the ratio between the moment of inertia of the whole pendulum about the supporting axis, and its moment about the same axis.*

If the slope of the lines of equation (34) be tan φ_1 and tan φ_2 then

$$\tan \varphi_1 = \frac{\pi^2 R}{g(b + Rd)} = \frac{2\pi^2 m}{gz(M + 2m)},$$

and

$$\tan \varphi_2 = \frac{\pi^2}{g} \frac{R}{L(R+1) - Rd - b} = \frac{\pi^2}{g} \frac{2m}{(L - z)(M + 2m)}.$$

In order that the intersection of the lines be as well defined as possible, the angle $(\varphi_1 - \varphi_2)$ must be as large as possible. So long as at least one of the angles is small this condition may be stated as (tan φ_1 − tan φ_2) as great as possible. This leads to the expression:

$$\tan \varphi_1 - \tan \varphi_2 = \frac{2\pi^2 m}{g(M + 2m)} \left(\frac{1}{z} - \frac{1}{L - z} \right), \tag{36}$$

This means m large, M small, *i. e.*, R large, z small and L large. Under these conditions (36) practically reduces to

$$\tan \varphi_1 - \tan \varphi_2 = \frac{\pi^2}{g} \times \frac{1}{z}$$

and is therefore inversely proportional to z. Now approximately $z = b/R + d$, and the above condition is seen to be that R shall be

large, b and d small and one of them negative: it being remembered that z must remain finite and positive.

Such a pendulum would consist of a bar as light as rigidity would permit, carrying two knife-edges and having two equal movable masses. Such a pendulum is shown in Fig. 5 and data derived from

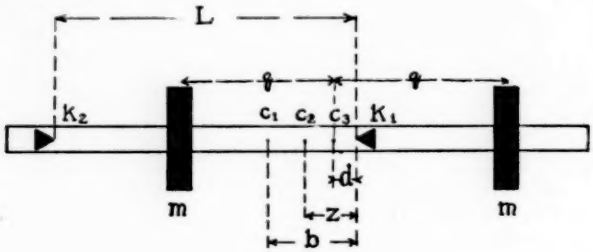

Fig. 5.

it illustrated in Fig. 6. In the lower diagram of this figure the intersection of the two lines is obtained by exterpolation and by using a pendulum so proportioned that it is impossible to bring it into final adjustment. It is a curious fact that if the graphs of the hyperbolæ themselves had been plotted (as in the first division of

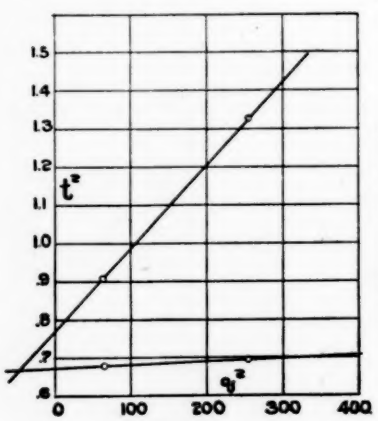

Fig. 6.

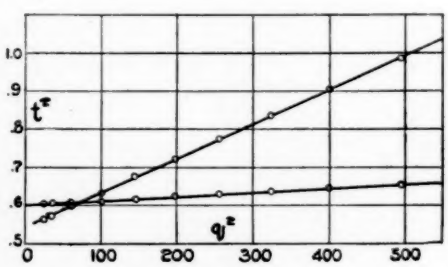

Fig. 6. a.

the paper) no intersection would have been secured. This is evident for the required value of q^2 is negative and q therefore imaginary. The curves would simply approach and recede as do curves

II. in Fig. 4. In spite of this fact the true value of g is gotten from the data, and the point of intersection in Fig. 6 is the means for getting this value.

Division III. t and R taken as variables.

If in equations (10) and (11) t and R be taken as the variables and the origin of (10) be moved to $(-b/d,\ 0)$ and of (11) to $[-(L-b)(L-d),\ 0]$ these equations reduce to:

$$\left.\begin{aligned} t_1^2 R' &= AR' + B \\ t_2^2 R' &= A'R' + B' \end{aligned}\right\}, \tag{37}$$

where

$$A = \frac{\pi^2}{g}\,\frac{d^2+\beta^2}{d}, \qquad B = \frac{\pi^2}{g}\,\frac{b}{d}\left(\frac{b^2+a^2}{b} - \frac{d^2+\beta^2}{d}\right),$$

$$A' = \frac{\pi^2}{g}\,\frac{(L-d)^2+\beta^2}{L-d}, \qquad B' = \frac{\pi^2}{g}\,\frac{L-b}{L-d}\left(\frac{(L-b)^2+a^2}{L-b} - \frac{(L-d)^2+\beta^2}{L-d}\right).$$

If the constants be positive these equations belong to Newton's sixtieth species of cubics. If B or B' be negative while the coeffi-

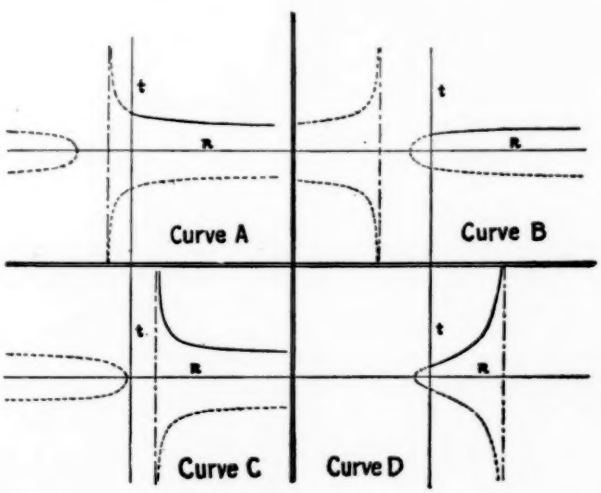

Fig. 7.

cient of R' is positive, changing R' to $-R'$ will have the effect of changing the sign of the absolute term, thus reducing the equa-

tion to the same species as before. If A or A' be negative, the corresponding equation will belong to Newton's sixty-third species, after changing R' to $-R'$ if necessary, to make the absolute term positive

In equation (10) four forms arise (Fig. 7). (**A**) b and d positive and $(b^2+\alpha^2)/b > (d^2+\beta^2)/d$. This makes A and B positive. (**B**) b and d positive, and $(b^2+\alpha^2)/b < (d^2+\beta^2)/d$. This makes A positive and B negative. (**C**) b negative and d positive. This makes both A and B positive. (**D**) b positive and d negative. This makes A and B both negative. Case (**A**) is distinguished from (**C**) in that the asymptote $R = -b/d$, to equation (10) is on the negative side of the Y-axis in (**A**) and on the positive side in (**C**). The values b and d negative, gives two more cases, which, however, do not concern us as no part of the resulting curves lies in the first quadrant. The four cases are illustrated in Fig. 7, where the general shape of the resulting curves is shown. Equation (11) yields a similar number of cases by substituting $L-b$ for b and $L-d$ for d.

Differentiating (10) and (11) with respect to R we obtain,

$$\left.\begin{aligned}\frac{dt}{dR} &= \frac{\pi}{2\sqrt{g}}\,\frac{(d^2+\beta^2)b-(b^2+\alpha^2)d}{(Rd+b)^{\frac{3}{2}}[(d^2+\beta^2)R+b^2+\alpha^2]^{\frac{1}{2}}},\\ \frac{dt'}{dR} &= \frac{\pi}{2\sqrt{g}}\,\frac{[(L-d)^2+\beta^2](L-b)-[(L-b)^2+\alpha^2](L-d)}{[R(L-d)+L-b]^{\frac{3}{2}}\{[(L-d)^2+\beta^2]R+(L-b)^2+\alpha^2\}^{\frac{1}{2}}}.\end{aligned}\right\}\quad(38)$$

The form of these derivatives shows that the curves have no maxima or minima. If b and d are both positive, dt/dR is always positive or always negative according as $(d^2+\beta^2)/d >$ or $< (b^2+\alpha^2)/b$. When

$$R=0,\quad t=\pi\sqrt{\frac{b^2+\alpha^2}{gb}}.$$

When

$$R=\infty,\quad t=\pi\sqrt{\frac{d^2+\beta^2}{gd}}.$$

The corresponding conclusions are reached for dt'/dR by putting $L-b$ for b and $L-d$ for d.

Equation (21) gives the lines passing through the intersection points, as in Division II., and the following are the coördinates of the intersections. From the first factor :

$$\left.\begin{aligned} t_1 &= \pm\pi\sqrt{\frac{L}{g}}, \\ R_1 &= -\frac{Lb-(b^2+a^2)}{Ld-(d^2+\beta^2)}. \end{aligned}\right\} \tag{39}$$

From the second factor:

$$\left.\begin{aligned} t_2 &= \pm\frac{\pi}{\sqrt{g}}\left[\frac{(L-2d)(b^2+a^2)-(L-2b)(d^2+\beta^2)}{L(b-d)}\right]^{\frac{1}{2}} \\ R_2 &= -\frac{L-2b}{L-2d}. \end{aligned}\right\} \tag{40}$$

It is here seen that while t is double valued (the + value alone being available) R is single valued in both cases. There are thus but two intersection points that can fall within the first quadrant.

Of course, the condition realized by the first intersection is the one desired, since here equation (22) holds good. To make it possible to obtain this intersection R_1 must be positive, and the following table gives all combinations of the signs of the four quantities b, $L-b$, d, $L-d$, with which it is possible to effect this. To the right of each combination are the necessary accompanying conditions on the constants. It will be found that all other combinations are either mere duplicates formed by interchanging the notation with respect to the knife-edges, or else are impossible because L and R are necessarily positive.

	b	$L-b$	d	$L-d$	
I.	+	+	+	+	$\frac{b^2+a^2}{b} > L > \frac{d^2+\beta^2}{d}$
II.	+	+	+	+	$\frac{b^2+a^2}{b} < L < \frac{d^2+\beta^2}{d}$
III.	+	+	+	−	$L > \frac{b^2+a^2}{b}$
IV.	+	−	+	+	$L > \frac{d^2+\beta^2}{d}$

Combination I. produces curve A (Fig. 7) from both knife-edges; II. gives curve B from both knife-edges; III. gives curves B and D from k_1 and k_2 respectively; IV. gives curve A from k_1 and curve B from k_2.

Conditions of the second intersection may be realized by any combination provided it is possible by a change in R to throw the centroid of the whole pendulum midway between k_1 and k_2. Since a tangency point, or conditions approaching it are preferably avoided, it would seem likely that a better defined first intersection would be obtained by rendering the second intersection impossible. This may be secured by adjusting the constants so that R_2 in (40) will be negative. The value of R_2 may be written

$$R_2 = -\frac{(L-b)-b}{(L-d)-d}, \tag{41}$$

a form that shows the symmetrical occurrence of the four determining quantities. In cases I. and II., where these quantities are all positive, in order to have R_2 negative $\frac{1}{2}L$ must be either greater or less than both b and d. If $\frac{1}{2}L$ be less than both, the conditions under case I. give $d > \beta$ and $L - b < a$; if $\frac{1}{2}L$ be greater than both, $b < a$ and $L - d > \beta$, for a positive first intersection. In case II, if $\frac{1}{2}L$ be less than b and d, we have $b > a$ and $L - d < \beta$; if $\frac{1}{2}L$ be greater, then $d < \beta$ and $L - b > a$. In case III. the necessary and sufficient condition for R_1 positive and R_2 negative is

$$2b > L > \frac{b^2 + a^2}{b}.$$

This shows that we must have $b > a$ and consequently $b > \frac{1}{2}L > a$, a simpler condition but not so precise as the preceding. The condition in case IV. is

$$2d > L > \frac{d^2 + \beta^2}{d}$$

from which may be deduced the simpler condition $d > \frac{1}{2}L > \beta$.

If we place the values of R_1 and t_1 from (39) in the expressions for the slopes of the two curves, given in (38) we get

$$\frac{dt_1}{dR_1} = \frac{\pi}{2\sqrt{Lg}} \frac{(a^2 + \beta^2 - Ld)^2}{b(d^2 + \beta^2) - d(b^2 + a^2)}, \tag{42}$$

$$\frac{dt_1'}{dR_1} = \frac{\pi}{2\sqrt{Lg}} \frac{(d^2 + \beta^2 - Ld)^2}{(L-b)[(L-d)^2 + \beta^2] - (L-d)[(L-b)^2 + a^2]}. \tag{43}$$

These expressions will be either both positive or both negative under the conditions we are discussing, hence, in general, to get a good angle of intersection it is well to make one as large, and the other as small, as practicable. To make an approximation we may make the denominator of (42) equal to zero, though in practice it should not exactly equal zero as then the curves break down into straight lines and the point of intersection R_1 becomes indeterminate. Doing this, the denominator of (43) reduces to

$$L[(b-d)(L-b-d)-a^2+\beta^2].$$

This should be large in order to make (43) small. To attain this we must evidently have the difference between b and d, and also the difference between a and β, large.

The above analysis, while reducing the number of permissible cases to four, does not furnish a sufficient criterion between them.

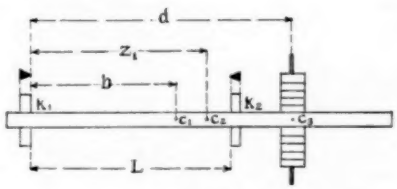

Fig. 8.

It is seen that any of the four may be realized experimentally and it would seem best to leave the question as to which (if any) of the four is to be preferred to be decided by experiment. A design for a pendulum meeting the requirements is given in Fig. 8. R is to be varied by adding thin, circular plates of metal to the cross-piece. It will be noticed that by using this form β is kept constant, as lengthening the cylinder does not alter its radius of gyration about its own axis.

We wish to acknowledge the assistance of Mr. Wm. N. Birchby, particularly in the preparation of the third division. Our thanks are also due to Mr. M. D. Hersey for the preparation of the diagrams.

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COLORADO COLLEGE,
December 1, 1906

MEASUREMENT OF ELECTROLYTIC RESISTANCE.

BY W. S. FRANKLIN AND L. A. FREUDENBERGER.

THE following methods of measuring electrolytic resistance are based on the use of a transformer, the primary of which is supplied with alternating current and whose secondary consists of a single "turn" of a liquid, the resistance of which is to be measured. The liquid is contained in a glass ring vessel, the ring of glass being formed of two concentric cylinders with the ends ground plane and covered with two flat glass plates. The core of the transformer is built up of 14 mil transformer iron, and it passes through the center of the ring-vessel, so that when the vessel is filled the iron core is surrounded by a closed turn of liquid of regular cross-section.

It was realized early in the work that an alternating current galvanometer was needed, of as great a sensibility as possible, considerably greater than a telephone receiver. For this purpose an alternating current galvanometer was devised by the authors.[1] The galvanometer may be described in brief as an ordinary Kelvin astatic galvanometer in which the suspension is replaced by a suspension consisting of two soft iron magnets inclined 45° to the axis of suspension, and so placed that when the center lines of the magnets are produced they meet in a common point. Such a magnetic system when placed in a uniform vertical alternating field, will be astatic, and by inclining the field slightly from the vertical a directive tendency may be given to the magnet system. If the coils of the Kelvin instrument are now supplied with alternating current of the same frequency as the vertical field, a uni-directional torque acts on the needle.

The first arrangement tried was as shown in Fig. 1.

The two primary coils, P and P', were connected in series across 110 volts 133 cycle mains; producing, with open secondaries, a

[1] See "A New Type of Alternating Current Galvanometer of High Sensibility," PHYS. REV., Vol. 24, No. 1, Jan., 1907.

magnetic flux in the transformer, as shown by the dotted line. The reluctances of the various legs of the transformer are adjusted so that no flux passes through the test coil, M, which is connected to a telephone receiver or alternating current galvanometer. If now a a load is thrown on the transformer by filling the glass vessel with an electrolyte, the balance of magnetomotive forces is disturbed, some flux crosses through the test coil, M, producing sound in the telephone receiver. If the other secondary coil is now closed

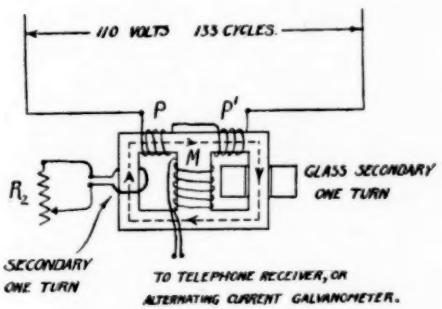

Fig. 1.

through an adjustable resistance, R_2, a balance of magnetomotive forces may again be obtained, and no sound heard in the telephone receiver; then R_2 is equal to the resistance of the electrolyte in the glass vessel. Of course, the load thrown on the transformer by the single turn of electrolyte is excessively small. It was found, for instance, that when $R_2 = 1$ ohm a deflection of 4 cm. (scale divisions) was produced at the galvanometer, from the zero position of the galvanometer (open-circuited secondaries). The length and cross-section of the electrolytic vessel was such as to make its resistance in ohms equal to the specific resistance of the electrolyte. Thus the presence of the highest conductivity electrolyte could just have been detected. The sensibility of galvanometer was 10^{-6} amperes per centimeter of scale division. Res. = 20 ohms. The galvanometer could not be used for smaller deflections as the zero-point fluctuated through a range of about one centimeter. The amount of flux crossing through the test coil then is extremely small, and, owing to the very low density, the permeability of the iron core through the test core must be extremely low; so low, in fact, as to act as though the iron were nearly a non-magnetic material.

The arrangement shown by Fig. 2 was next tried, in which the coils of the galvanometer were wound differentially.

With open secondaries, and two exactly similar transformers, the slider, P, may be adjusted to give zero deflection of the differential galvanometer. The primary coils were connected in series across 30 volt 133 cycle mains. In this, as well as in the first arrangement, the number of turns on the primary coils was such

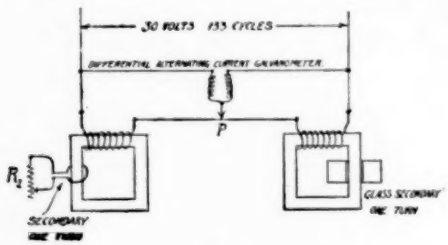

Fig. 2.

as to work the iron to a density of about 1,000 lines per square centimeter. On filling the glass vessel with an electrolyte, the balance on the differential galvanometer is disturbed and may be restored by connecting a resistance, R_2, across the other secondary coil, equal in value to the resistance of the electrolyte.

With the transformers balanced so as to give zero deflection of the differential galvanometer, with open secondaries, a resistance of 10 ohms (R_2) gave a deflection of 2 centimeters (scale divisions). The sensibility of the differential galvanometer was 10^{-6} amp. per centimeter of scale deflection. Res. of instrument = 1,000 ohms. Some sensibility had to be sacrificed in the galvanometer as it was found necessary to wind the coils of the galvanometer with large clearance between the coils and the suspended magnets ; otherwise the differential action of the two windings of the instrument was dependent upon the position of the magnets. The variability of zero was again at least one centimeter of scale deflection.

In both the above methods great difficulty was experienced in properly insulating the coils on the transformers from each other and from the iron core. It was found necessary to insulate with glass. No great attempt was made to attain maximum sensibility in either of the previous methods as they were abandoned in favor

of the following method, shown by Fig. 3, which represents a Wheatstone bridge in which the electrolyte is connected electromagnetically to one of the arms of the bridge.

A slide wire, AB, consisting of 120 feet of No. 16 B. & S. German Silver wire, is wound back and forth on a flat wood surface of convenient size. With secondaries open-circuited the slider, P, is adjusted for zero galvanometer deflection. With the galvanometer

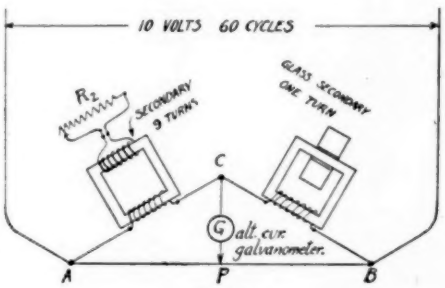

Fig. 3.

used (20 ohms) and a sensibility of 10^{-6} amperes per centimeter of scale deflection, a movement of P of one sixteenth inch in the 120 feet of slide wire caused a galvanometer deflection of 10 centimeters (scale divisions). With a telephone receiver the balance point on the slide wire could only be located within six inches.

The secondary coil actually used consisted of nine turns (instead of one turn) so that the resistance (R_2) necessary to balance the electrolyte of one turn must be divided by 81 (the square of the ratio of turns) to obtain the actual resistance of the electrolyte contained in the glass vessel.

The slider, P, being set so as to give zero galvanometer deflection, 10,000 ohms (R_2) produced a deflection of 2 cm. (scale divisions). The maximum variation of zero, throughout tests was about 1 cm., with an average variation of about .5 cm. A commercial lighting circuit of rather poor regulation was used as the source of current, and the zero of the galvanometer shifted rapidly within the limits specified. With a telephone receiver a resistance of $R_2 = 100$ ohms could just be detected.

The dimensions of the glass ring-vessel were as follows: The inside dimensions of course being given, corresponding to the dimensions of the electrolyte.

Inside diameter	7.83 cm. $= r_1$
Outside diameter	19.60 " $= r_2$
Breadth	6.00 " $= b$.

The above dimensions were obtained by averaging the dimensions around the ring. The dimensions given did nor differ more than 0.01 cm. in going around the ring.

The specific conductivity, c, of an electrolyte in reciprocal ohms per centimeter cube is

$$c = \frac{81 \times 1.1413}{R_2}$$

in which R_2 is the balancing resistance shown in Fig. 3 and 1.1413 is the conductivity constant of the given glass cell.

The volume of the electrolytic vessel (by calculation) was 1,521.4 cu. cm. The measured volume (by filling with water) was $1,520 \pm 2$ cu. cm. Owing to the large size of the electrolytic vessel and the regularity of its dimensions, the conductivity constant of the cell can certainly be determined with greater accuracy by calculation from measured dimensions than by measuring its resistance when filled with mercury.

The two primary coils (Fig. 3) were exactly alike and form-wound on glass having 30 turns each; the secondary coil of 9 turns was wound to the same cross-section as the glass vessel; the effective section of iron in each transformer was 17.8 sq. cm., producing a maximum flux density in the iron of about 3,000 lines per sq. cm. Each sheet of 14 mil transformer punching was shellacked and insulated by paper to reduce eddy current losses. By calculation, the magnetizing current of transformer was 1.83 amperes, and the current supplying hysteresis and eddy current losses was 0.178 amp. The no-load current of the transformers thus lags about 84°.5 behind the impressed e.m.f. in phase. Each transformer may then be considered to consist of a pure reactance of 2.7 ohms in parallel with a pure resistance of 28.0 ohms. When the secondary turn of glass is filled with one of the highest conductivity electrolytes (H_2SO_4 30 per cent.) its resistance is about 1 ohm, or reduced to the 30 turn primary it is about 900 ohms. In measuring the resistance of the electrolyte, we are thus measuring the *change* of current produced by connecting a resistance of 900 ohms in parallel

with an impedance of about 2.7 ohms. From the following results it may be seen that this small change of current has been measured with an accuracy of about .5 per cent. for H_2SO_4. The equivalent resistance and reactance of each primary circuit is not perfectly constant, as small variations are produced in their values due to fluctuations in line e.m.f. and frequency, and also due to temperature differences in each transformer, the zero point of the galvanometer changed continuously in a uniform manner. Since it requires several minutes to fill and empty the glass vessel, the uniform shift of the zero was allowed for by interpolation; keeping current on the transformers for about half an hour until they settled to fairly uniform conditions.

In working the galvanometer the vertical field was brought nearly in phase with the current in the galvanometer (produced by the e.m.f.; *PC*, Fig. 3) by varying the phase position of the current producing the galvanometer field. Any large phase difference (approaching 90°) between the field of galvanometer and the galvanometer current would, of course, considerably reduce the sensibility of the instrument, but would not introduce any errors in the measured resistances.

SAMPLE SET OF OBSERVATIONS.

30 *per cent. solution of* H_2SO_4 *at* 19° *C.*

		1.	2	3	4
I.	R_2 = infinity. Glass vessel 0.7 full.	34.8 27.5	+ 7.3		28m 30s
II.	R_2 = 120 ohms. Glass vessel full.	40.9 23.9	+17.0	+ 8.9	30 15
III.	R_2 = 140 ohms. Glass vessel full.	28.7 31.3	− 2.6	+ 9.3	30 45
IV.	R_2 = infinity. Glass vessel 0.7 full.	35.0 25.0	+10.0		31 30
V.	R_2 = infinity. Glass vessel empty.	39.2 21.0	+18.2	+10.7	32 15

Below are given the observations for a low-resistance electrolyte. It was found necessary to make a small correction for eddy currents in the electrolyte. That is, after determining the galvanometer zero with the glass vessel empty; then, on filling to 0.7 full so that the turn of liquid was still broken, the zero point of the galvanom-

eter had shifted. The galvanometer zero at 0.7 full of liquid was used as the reference point, and the correction for eddy currents for the remaining 0.3 volume could then be interpolated. Double deflections were observed by a reversing switch (always thrown in one direction) in the galvanometer circuit.

The numbers given in column I. are the galvanometer readings corresponding to the two positions of the reversing switch in the galvanometer circuit, the numbers in column 2 are the double galvanometer deflections, the numbers in column 3 are the interpolated zero points of the galvanometer based on observations I. and IV., and the numbers in column 4 are the mean clock readings of each pair of galvanometer readings given in column 1. The drift from I. to IV. being known, the zero point for II., III., and V. can be interpolated quite accurately, since the drift was found to be very nearly linear with respect to time.

Readings II. and III. serve to determine the value of R_2 (neglecting the effect of eddy currents). That is

(II.) $R_2 = 120$ ohms gives deflection from $+8.9$ to $+17.0 = 8.1$ right.

(III.) $R_2 = 140$ ohms gives deflection from $+9.3$ to $-2.6 = 11.9$ left.

Therefore $R_2 = 128.1$ ohms.

Since R_2 has been determined from galvanometer deflections which include the deflection caused by eddy currents due to 0.7 volume of liquid, it remains, from readings IV. and V., to correct for the eddy currents due to 0.3 volume of liquid. From (IV.) and (V.) it is seen that *emptying* the vessel from 0.7 full produces a deflection from $+10.7$ to $+18.2 = 7.5$ right. *Filling* from 0.7 full would therefore cause a deflection of $0.3/0.7 \times 7.5$ left $= 3.2$ cm. which is equivalent to 6.4 ohms [see (II.) and (III.) above].

This correction, occurring on transformer, CB (Fig. 3), means an opposite correction on the other transformer, AC. Thus the correction when referred to R_2 is 3.2 cm. *right*, which is equivalent to 6.4 ohms in the direction of lower values of R_2. Thus the corrected value of R_2 is $128.1 - 6.4 = 121.7$ ohms, and the specific conductivity $= C = 81 \times 1.1413/121.7 = 0.759$ reciprocal ohms per centimeter cube.

The temperature of the electrolyte was measured immediately before filling the glass vessel and immediately after emptying. Temperatures could scarcely have been determined, however, closer than about 0°.1 C. As the temperature correction for 0°.1 C. affects the third place in the specific conductivity, any attempt to secure greater accuracy would necessitate submerging the trans-

TABLE OF RESULTS.

Solution.	Percentage by Weight.	Specific Gravity at 15.5° C.	Temperature Degrees Centigrade.	No. of Observations.	Conductivity in Reciprocal Ohms per Centimeter Cube.	Conductivity (Kohlrausch).	Temperature Coefficient at 18° C.	Temperature Coefficient (Kohlrausch).
KCl.	19.7	1.1327	12.55	3	0.232 ± .005	0.2394	0.017	0.0169
			25.38	4	0.290 ± .002	0.2967		
	15	1.1000	12.53	4	0.184 ± .007	0.1822	0.015	0.0179
			20.67	5	0.209 ± .003	0.2116		
	10	1.0657	22.50	7	0.140 ± .003	0.1474		
	5.25	1.0345	12.44	4	0.074 ± .005	0.0643	0.014	0.0200
			21.50	5	0.084 ± .004	0.0775		
NaCl.	24.6	1.1865	8.36	11	0.171 ± .004	0.1659	0.022	0.0226
			18.49	2	0.212 ± .001	0.2121		
			22.25	4	0.237 ± .001	0.2325		
	15	1.1120	13.20	4	0.141 ± .008	0.1475		
KNO_3.	9.52	1.0620	15.57	5	0.082 ± .004	0.0762		
$NaNO_3$.	30	1.2277	23.19	5	0.174 ± .003	0.1798		
KOH.	35.7	1.3570	18.10	7	0.501 ± .007	0.5018	0.029	0.0283
			21.95	6	0.558 ± .012	0.5495		
	28.56	1.2800	22.00	6	0.584 ± .004	0.5903		
	21.42	1.2060	23.00	6	0.553 ± .005	0.5650		
	14.28	1.1410	21.90	5	0.445 ± .004	0.4383		
	7.14	1.0695	20.50	7	0.256 ± .004	0.2454		
	3.57	1.0340	17.80	5	0.129 ± .004	0.1270		
H_2SO_4.	95.6	1.8385	17.10	5	0.108 ± .006	0.0963		
	60	1.4965	19.50	4	0.402 ± .004	0.3846		
	30	1.2200	19.00	4	0.730 ± .010	0 7508	0.013	0.0162
			28.83	5	0.820 ± .025	0.8684		
	20	1.1420	19.32	4	0.639 ± .010	0.6651		
	10	1.0675	19.81	4	0.383 ± .002	0.4005		
	5	1.0330	19.71	6	0.216 ± .007	0.2128		

NOTE.—In the sixth column (of conductivity), the uncertainty is given as the average deviation from the mean.

former in a thermostat, which would no doubt materially help in reducing the drift of the zero point of the galvanometer, due to unequal temperature changes in the iron of the two transformers.

The foregoing table contains the results of measurements on a number of solutions of salts, bases, and acids of varying percentage strengths. Kohlrausch's values (see "Das Leitvermogen der Electrolyte" by Kohlrausch and Holborn, 1898) for the same concentrations and temperatures are given for comparison.

An accurate Wheatstone bridge was used as a rheostat for R_2 (Fig. 3). The bridge was then compared with a new bridge (certified by the Reichsanstalt) and the values of resistances found to be correct within .04 per cent.

An inspection of the above table will show a close agreement, within experimental error, between the results here obtained by the electrodeless ring method and the results of Kohlrausch and Holborn.

The apparatus for the above investigations was constructed under a grant from the Carnegie Institution of Washington, and the experimental work has been done chiefly by the junior author.

NEW BOOKS.

The Theory of Optics. By A. SCHUSTER. London, E. Arnold.
Lehrbuch der Optik. Von PAUL DRUDE. Leipzig, S. Hirzel.
Physical Optics. By R. W. WOOD. New York, The Macmillan Co.

Schuster's Introduction to the Theory of Optics is avowedly influenced by a desire to retain the older philosophical notion of the province of a physical theory. Whether one agrees or not with the author's objection to the "evasive school of philosophy which has received some support from the writings of Heinrich Hertz" it will, in general, be admitted that the treatment is an able one. There is greater continuity of treatment than in some recent works, and because of this, as well as its limitation to topics more frequently discussed in text books, will doubtless commend itself to many for class room use. The discussion of interference, of Talbot's bands and of the nature of white light as well as the chapter on the transmission of energy are worthy of special mention. The biographical notices of those eminent in the development of optics are interesting breaks in the rather rigorous text. For those who incline to a deductive and rather strictly formal treatment this book will be especially pleasing. Many may be more attracted by Wood's comprehensive survey of the experimental field even if the demands of "philosophical" exactness are not so fully met. The book is an admirable example of the printers' art and is remarkably free from typographical errors.

The revised edition of Drude's Optics is of special interest as the last work of this brilliant physicist whose untimely death is mourned by the entire physical world. The revision contains some considerable extensions of the electron hypotheses to the explanation of dispersion, magneto-optics and to radiation. A very clear resumé of Planck's radiation formulæ and their consequences is given and several new references are added in various parts of the work. Had Drude given to science nothing besides this text book with its numerous examples of his genius both as an experimenter and a mathematician his place would have been firmly fixed among the world's truly great physicists.

For clearness and breadth, for depth and suggestiveness, for balance between experiment and theory, this text may well stand as a monument to one of the greatest of modern scientists. Surely all English speaking men of science will gladly join in the wish expressed so well by his assistant Dr. Kiebitz. Möge diesem Buch aussen seinem wissenschaftlichen

Inhalt auch Drude's glückliche Art, wissentschaftliche Probleme aufzufassen und zu behandeln, bleibenden Wert verliehen, und möge es jeden Leser etwas von dem Geiste empfinden lassen, der ihn zu seiner Lehrtatigkeit begeistert hat, der ihn den Giffel wissenschaftlichen Forschers und Erkennens in der Veredelung der Lebensauffssung und Lebensfuhrung erblicken liess.

In Wood's Physical Optics we have the emphasis placed on the experimental side of the subject. It would be difficult to collect a more instructive and interesting group of experiments in optics than that presented. The strictly mathematical portions of the book strikes one as somewhat disconnected and resembles a collection of rather carefully selected notes on the subject rather than a formal development. Not that such a treatment is undesirable but that it is in decided contrast with Schuster's or Drude's presentation. The present problems in physical optics are very clearly indicated and many extensions of our knowledge are suggested. Many interesting explanations of rather complicated solar phenomena are given. The work is marred somewhat by occasional errors and misprints. For example, on p. 165, Fig. 141, the axial figures should each have one dimension twice the size of those not on the axis [an error quite common in text books]. On page 117, $x = (\mu - 1)\varepsilon\, a/5$, s should replace the 5. In quite a number of places the notation does not agree with the figures. Apart from these slight defects the book is an inspiration to students and teachers and will be a great aid in rescuing physical optics from the absurd mathematical symbolism which sometimes seems to throttle progress in this fruitful field of investigation.

With these three valuable books at disposal we are better provided with introductions to optical theory than to most other branches of physics.

J. S. SHEARER.